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# Probability Techniques for Reliability Analysis of Composite Materials

Robert C. Wetherhold and Anthony M. Ucci  
*State University of New York*  
*Buffalo, New York*

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# PROBABILITY TECHNIQUES FOR RELIABILITY ANALYSIS OF COMPOSITE MATERIALS

Robert C. Wetherhold and Anthony M. Ucci  
Department of Mechanical and Aerospace Engineering  
State University of New York  
Buffalo, New York 14260-4400

## SUMMARY

Traditional design approaches for composite materials have employed deterministic criteria for failure analysis. New approaches are required to predict the reliability of composite structures, since strengths and stresses may be random variables. This report will examine and compare methods used to evaluate the reliability of composite laminae. The two types of methods that will be evaluated are fast probability integration (FPI) methods and Monte Carlo methods. In these methods, reliability is formulated as the probability that an explicit function of random variables is less than a given constant. Using failure criteria developed for composite materials, a function of design variables can be generated which defines a "failure surface" in probability space. A number of methods are available to evaluate the integration over the probability space bounded by this surface; this integration delivers the required reliability. The methods which will be evaluated are: the first order, second moment FPI methods; second order, second moment FPI methods; the simple Monte Carlo; and an advanced Monte Carlo technique which utilizes importance sampling. The methods are compared for accuracy, efficiency, and for the conservatism of the reliability estimation. The methodology involved in determining the sensitivity of the reliability estimate to the design variables (strength distributions) and importance factors is also presented.

## 1. INTRODUCTION

### Composite Materials

Fibrous composite materials consist of at least two constituents: a series of purposefully oriented fibers, surrounded by a solid matrix. Typically, the fibers act as load-carrying members while the matrix transfers the load between them while fixing the fibers in the desired orientation and location within the composite. The resulting material is both strong and stiff. In addition to improving strength and stiffness, the mixture improves the survivability of the material by improving toughness and protecting the usually susceptible fibers from environmental damage. The most common form of composites is a stack of relatively thin layers of fibers and matrix (laminae), each with a specific fiber orientation, bonded together to form a laminate. The orientations and the stacking sequence (of laminae) can be controlled in order to generate a variety of mechanical properties.

Due to the relatively low density of the matrix material, composites usually have much higher strength-to-weight ratios and stiffness-to-weight ratios when compared to traditional structural materials (e.g., metals). In addition, composites are often more fatigue resistant than these materials. Unlike traditional materials, which have nearly equal properties regardless of the direction of measurement, composites frequently have anisotropic material properties. In most unidirectionally oriented fiber-reinforced composites, the tensile strength and modulus are a maximum in the longitudinal or axial direction of the fibers. Many other properties, such as thermal and moisture expansion, thermal conductivity and impact strength are also directionally dependent. Most laminates use a multidirectional stacking sequence to provide a desired or tailored set of material properties.

Although composites offer the possibility of tailoring material properties to a particular application, their design requires more care than monolithic metals or ceramics. The elastic properties of composites are relatively well controlled and reproducible, but the strength of composites may be more widely distributed due to the inherent inhomogeneity of the microstructure. It is sensible then, to speak not of "safety factors," but of reliability, using a knowledge of the distribution of composite strengths. We shall first review the use of deterministic failure theories, then indicate how a "design variable" approach can be used to calculate a reliability value.

## Review of Deterministic Failure Theories

Failure analysis in a design problem consists of the comparison of the stress (due to the applied load) to the resistance (or strength) of the material. With the application of multiaxial stresses, a failure theory based on the combined stresses must be used. In traditional materials (such as aluminum or steel) the theories commonly used are the maximum allowable shear stress (Tresca) theory or the distortional energy (von Mises criterion). These theories utilize the isotropy of the materials and are based on plastic deformation as the mode of failure. Composites are anisotropic and do not normally exhibit this type of failure. Therefore failure theories developed for isotropic materials are not applicable to composite materials. Many new failure theories have been developed specifically for composites; these rely on a variety of failure definitions. They also consider the directionality and geometry of the composite.

For application in a thin laminated structure, the strength of the lamina is characterized by five strength properties in the material's coordinate system:

- $S_{Lt}$  = Longitudinal (Axial) Tensile Strength
- $S_{Lc}$  = Longitudinal (Axial) Compressive Strength
- $S_{Tt}$  = Transverse Tensile Strength
- $S_{Tc}$  = Transverse Compressive Strength
- $S_{LTs}$  = In-plane (Interlaminar) Shear Strength

Both the longitudinal and transverse strength are sign-dependent, while the shear property is independent of the sign of the loading, given the usual orthotropic symmetry. These five properties must be determined experimentally.

For failure analysis the plane stress condition of an orthotropic lamina is assumed. This is shown in figure 1-1.

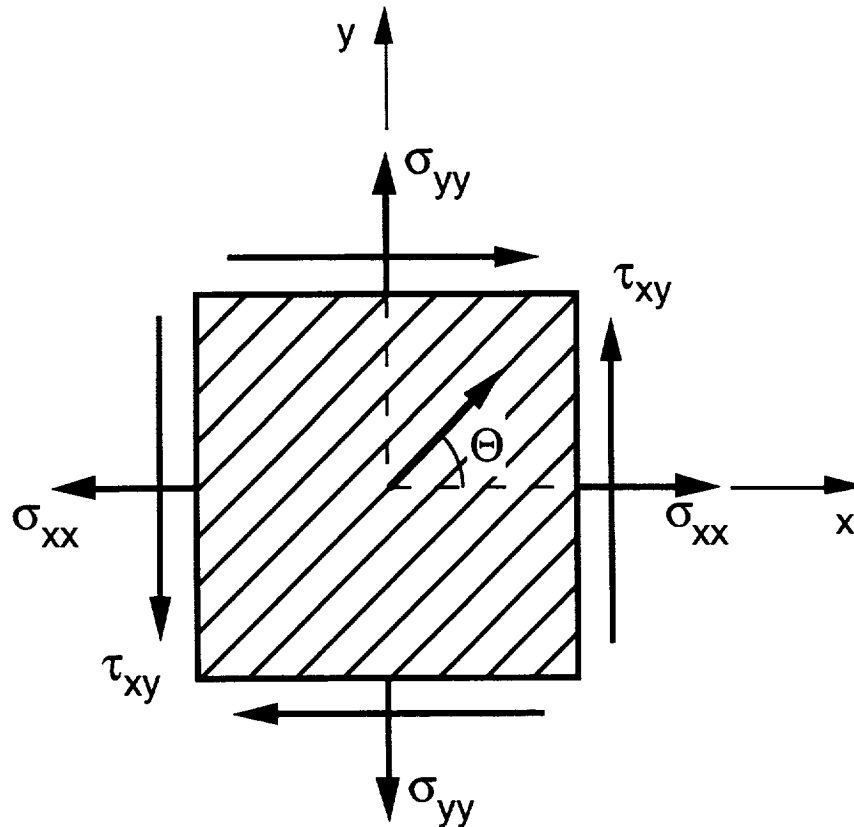


Figure 1-1: General Stress State of an Orthotropic Lamina [10]

The lamina contains unidirectional fibers at a fiber orientation angle of  $\Theta$  with respect to the primary geometric axis (x). The proposed failure theories require the transformation of the applied stresses from the lamina or geometric coordinate system (x, y) to the principal material coordinate system (1, 2). This transformation is defined by equations (1.1) to (1.3).

$$\sigma_{11} = \sigma_{xx}\cos^2\Theta + \sigma_{yy}\sin^2\Theta + 2\tau_{xy}\sin\Theta\cos\Theta \quad (1.1)$$

$$\sigma_{22} = \sigma_{xx}\sin^2\Theta + \sigma_{yy}\cos^2\Theta - 2\tau_{xy}\sin\Theta\cos\Theta \quad (1.2)$$

$$\tau_{12} = (\sigma_{yy} - \sigma_{xx})\sin\Theta\cos\Theta + \tau_{xy}(\cos^2\Theta - \sin^2\Theta) \quad (1.3)$$

where  $\sigma_{11}$ ,  $\sigma_{22}$  and  $\tau_{12}$  are the transformed stresses, longitudinal, transverse and shear respectively, and  $\sigma_{xx}$ ,  $\sigma_{yy}$  and  $\tau_{xy}$  are the applied stresses. The two failures theories used to evaluate reliability in this report are the Tsai-Hill theory and the more general Tsai-Wu theory.

**Tsai-Hill:** This theory, developed by V.D. Azzi and S.W. Tsai, and also referred to as the Azzi-Tsai-Hill failure theory, is based on the anisotropic yield criterion for metals (developed by Hill). It proposes that failure occurs when equation (1.4) is satisfied (ref. 10):

$$\frac{\sigma_{11}^2}{S_{Lt}^2} - \frac{\sigma_{11}\sigma_{22}}{S_{Lt}^2} + \frac{\sigma_{22}^2}{S_{Tt}^2} + \frac{\tau_{12}^2}{S_{LTs}^2} = 1 \quad (1.4)$$

This equality is for tensile loads only. If compressive loads are applied (either in the longitudinal or transverse direction) then the corresponding compressive strengths are substituted. This criterion defines a failure envelope which is the graphical representation of the failure theory in the stress coordinate system. The envelope forms the boundary between the the safe region and the failure region. Using  $\sigma_{11}$  and  $\sigma_{22}$  as the variables defining the coordinate axes the failure theory can be defined on a quadrant-by-quadrant basis.

In the  $+\sigma_{11}/+\sigma_{22}$  quadrant,

$$\frac{\sigma_{11}^2}{S_{Lt}^2} - \frac{\sigma_{11}\sigma_{22}}{S_{Lt}^2} + \frac{\sigma_{22}^2}{S_{Tt}^2} = 1 - \frac{\tau_{12}^2}{S_{LTs}^2} \quad (1.5)$$

In the  $+\sigma_{11}/-\sigma_{22}$  quadrant,

$$\frac{\sigma_{11}^2}{S_{Lt}^2} - \frac{\sigma_{11}\sigma_{22}}{S_{Lt}^2} + \frac{\sigma_{22}^2}{S_{Tc}^2} = 1 - \frac{\tau_{12}^2}{S_{LTs}^2} \quad (1.6)$$

In the  $-\sigma_{11}/+\sigma_{22}$  quadrant,

$$\frac{\sigma_{11}^2}{S_{Lc}^2} - \frac{\sigma_{11}\sigma_{22}}{S_{Lc}^2} + \frac{\sigma_{22}^2}{S_{Tt}^2} = 1 - \frac{\tau_{12}^2}{S_{LTs}^2} \quad (1.7)$$

In the  $-\sigma_{11}/-\sigma_{22}$  quadrant,

$$\frac{\sigma_{11}^2}{S_{Lc}^2} - \frac{\sigma_{11}\sigma_{22}}{S_{Lc}^2} + \frac{\sigma_{22}^2}{S_{Tc}^2} = 1 - \frac{\tau_{12}^2}{S_{LTs}^2} \quad (1.8)$$

These four equations define a failure envelope. The continuous but non-smooth nature of the failure envelope is due to the anisotropic strength characteristics.

**Tsai-Wu:** This failure theory (developed by S.W. Tsai and E.M. Wu) proposes that failure occurs when the following equality (equation (1.9)) is satisfied (ref. 10):

$$F_1\sigma_{11} + F_2\sigma_{22} + F_6\tau_{12} + F_{11}\sigma_{11}^2 + F_{22}\sigma_{22}^2 + F_{66}\tau_{12}^2 + 2F_{12}\sigma_{11}\sigma_{22} = 1 \quad (1.9)$$

where  $F_i$  and  $F_{ij}$  are called the strength coefficients and are defined as:

$$\begin{aligned} F_1 &= \frac{1}{S_{Lt}} - \frac{1}{S_{Lc}} & F_2 &= \frac{1}{S_{Tt}} - \frac{1}{S_{Tc}} & F_6 &= 0 \\ F_{11} &= \frac{1}{S_{Lt}S_{Lc}} & F_{22} &= \frac{1}{S_{Tt}S_{Tc}} & F_{66} &= \frac{1}{S_{LTs}^2} \end{aligned}$$

with  $S_{Lc}$  and  $S_{Tc} > 0$ . The  $F_{12}$  term represents the interaction of the longitudinal and transverse stresses. Since all other factors are known, this term can be determined by a number of experimental procedures. One possible technique is to perform an equi-biaxial test (with applied transverse and longitudinal stresses only), where  $\sigma_{11} = \sigma_{22} = \sigma$  at failure. In this case equation (1.9) reduces to:

$$(F_1 + F_2)\sigma + (F_{11} + F_{22} + 2F_{12})\sigma^2 = 1 \quad (1.10)$$

Using this equation and the definitions for the strength coefficients the strength interaction term  $F_{12}$  can be recovered:

$$F_{12} = \frac{1}{2\sigma^2} \left[ 1 - \left( \frac{1}{S_{Lt}} - \frac{1}{S_{Lc}} + \frac{1}{S_{Tt}} - \frac{1}{S_{Tc}} \right) \sigma - \left( \frac{1}{S_{Lt}S_{Lc}} + \frac{1}{S_{Tt}S_{Tc}} \right) \sigma^2 \right] \quad (1.11)$$

In order to ensure a closed failure surface, the following inequality must be satisfied:

$$-\frac{1}{2}(F_{11}F_{22})^{1/2} \leq F_{12} \leq \frac{1}{2}(F_{11}F_{22})^{1/2}$$

Due to the limited amount of experimental data available, the lower limit of this range will be used in this report, consistent with an analogy to the two-dimensional Von Mises criterion (ref. 14). These equations define the boundary between safe and failure regions. Unlike the envelope defined by the Tsai-Hill failure criterion, this failure envelope is continuous and smooth. This is due to the presence of linear terms in the failure equation.

The envelopes defined by these two failure theories are compared graphically in figures 1-2 for a typical carbon fiber/epoxy matrix composite.

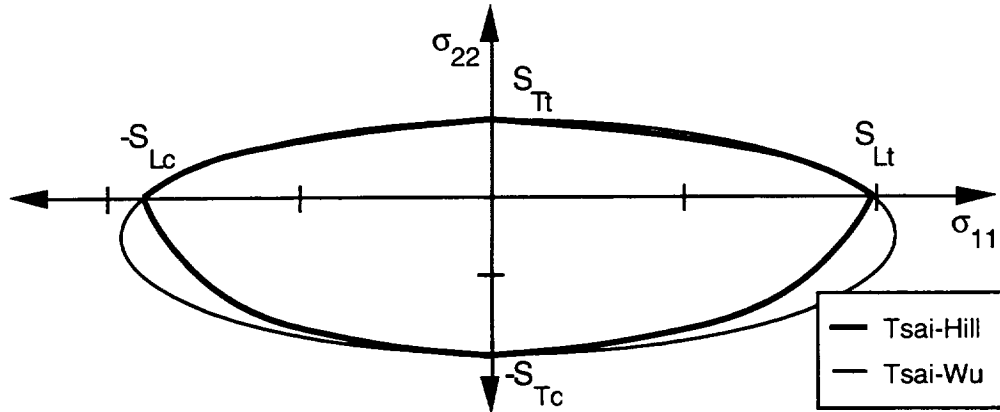


Figure 1-2: Tsai-Wu and Tsai-Hill Failure Theories[10]

Note that the scale of the  $\sigma_{22}$  axis has been exaggerated to show the curvature. The failure envelope defined by the Tsai-Wu failure theory is a continuous and smooth ellipse while the Tsai-Hill envelope is smooth within each quadrant, and only continuous at the interface between quadrants. Both envelopes intercept the  $\sigma_{11}$  axis at  $S_{Lt}$  and  $-S_{Lc}$ , and the  $\sigma_{22}$  axis at  $S_{Tt}$  and  $-S_{Tc}$ .

Application of these theories is limited to a deterministic analysis. Given a known set of stresses and strengths, these failure criteria can predict failure or survival. For the reliability analysis involved in a design problem, this is of little use. In most cases we are attempting to analyze the reliability of specimens of an unknown, but statistically describable, strength. The only true method to determine the strength for a given specimen would be to load that specimen to failure. This would give us an accurate measure of strength but in so doing would destroy the specimen. The reliability method desired would use deterministic stresses (we assume these can be accurately measured or calculated) with composite strengths following known distributions. These composite strengths would be obtained by testing specimens until the distributions of their strengths could be predicted with a degree of confidence. The reliability methods discussed in this report will use this information and the failure theories available to predict reliability.

### Design Space/Integration Method

To fully understand the methods used for calculating reliability, the term “design space” must be defined. This  $n$ -dimensional space (hyperspace), where  $n$  is the number of design variables, represents the entire domain of possible values of the variables. A joint density function, which predicts the relative probability of occurrence of a specific set of values for the design variables (coordinates in design space) is defined over this entire domain. The next step is to determine which areas of the design space, representing possible sets of design variables, will result in the failure of the system. Although we usually consider the strengths as design variables and the applied stresses as known values, it is straight-forward to add the stresses to the set of design variables.

In general, a failure criterion defines a multi-dimensional surface which divides the hyperspace into two domains; a safe region and a failure region. To determine the reliability, the defined joint density function is integrated over the safe region while conversely the probability of failure is the integral over the failure region. The integral of a density function is termed a (cumulative) distribution function. For a single variable system this analysis requires the integration of the single-variable density function over a defined range. Problems with multiple variables require the more complex (multivariate) joint density function. Consider the multivariate case with  $n=2$ . If the variables  $x$  and  $y$  are both normally distributed, then their joint density function decays continuously from the mean values ( $\mu_x, \mu_y$ ) and appears as a “pile of sand.” Figure 1-3 schematically shows this for two design variables (where  $\sigma_x = \sigma_y = \sigma$ , the standard deviation).

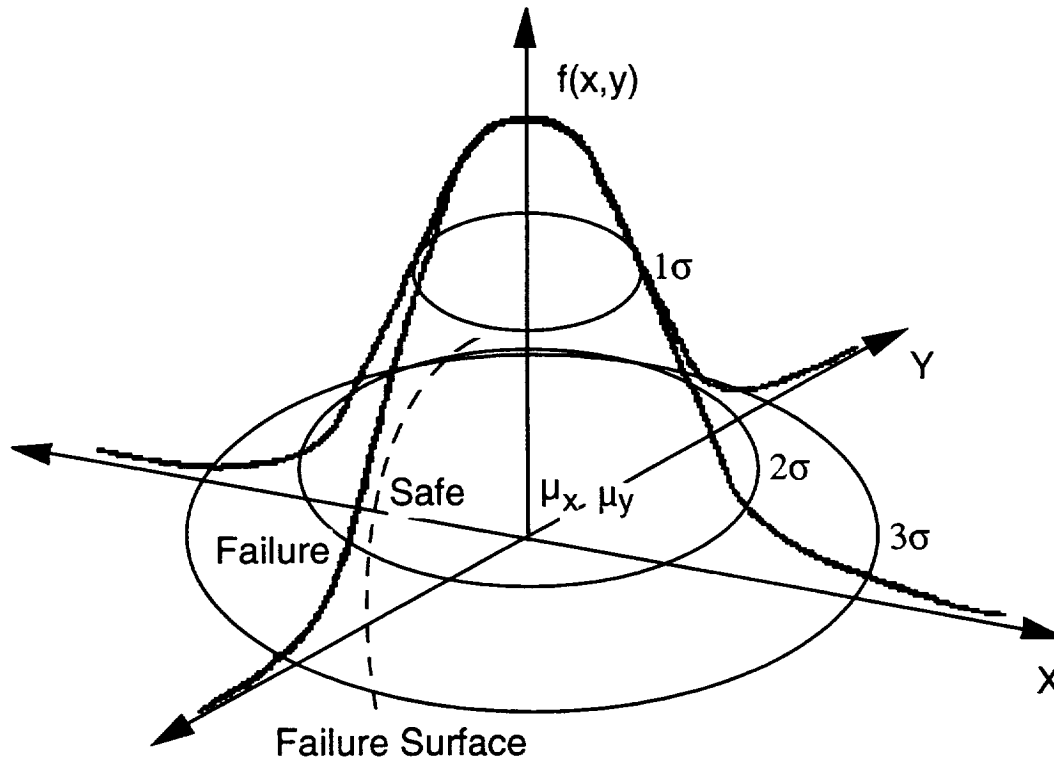


Figure 1-3: 2D Joint Density Function

For the failure criteria defined in the previous section, the design problem consists of three variables for the Tsai-Hill failure criterion and five variables for the Tsai-Wu criterion. The joint density functions associated with these variables are similar to the one shown in figure 1-3, but generalized to three or five dimensions. The first step in the integration of these density functions is to define the bounding surface for the safe region. Once this is accomplished, the density function is then integrated over the volume (area) contained within the surface. The complexity of the typical bounding surface defined by these failure criteria makes these integrals very difficult to evaluate in closed form. Fast Probability Integration (FPI) methods rely on approximating the failure surface by a predetermined geometric form. These geometric forms are chosen specifically to match analytical forms for which evaluation of the integral for the reliability is practical. In choosing an appropriate approximating geometric form, a compromise between accuracy and computational difficulty is made. Monte Carlo methods are also widely used, although they may require many simulation trials using a random sampling method to evaluate the integral.

In this report we will consider the application of both FPI and Monte Carlo methods to the calculation of reliability for composite materials under multiaxial loading. The composite materials considered are: a unidirectional fiber, polymer matrix composite (PMC) of carbon (graphite)/epoxy, and a ceramic matrix composite of silicon carbide (SiC)/reaction bonded silicon nitride (RBSN). The analysis will demonstrate the sensitivity of the calculated reliability to parameters of the strength distributions. Both normal and non-normal parameter distributions will be evaluated. In addition to this, a general comparison of the methods will be performed.

## 2. RELIABILITY ANALYSIS TECHNIQUES

The process involved in performing an integration to evaluate a probability of failure can be viewed in its simplest form as a design problem involving a single random variable. If the system involved has a deterministic resistance (strength),  $x$ , and a random applied load,  $Y$ , then the reliability of the system is the probability that the resistance is greater than the load,  $P(x > Y)$ . Conversely the failure probability is the

probability that the load is greater than the strength,  $P(Y > x)$ . To determine this probability the distribution of the load must be determined. First, define a probability density function by:

$$f_Y(y) dy = P[y \leq Y \leq y+dy]$$

Once the probability density function for the load,  $f_Y(y)$ , is known, an integration is performed over the unsafe region (where the load is greater than the strength) to determine the probability of failure. This is shown graphically in figure 2-1:

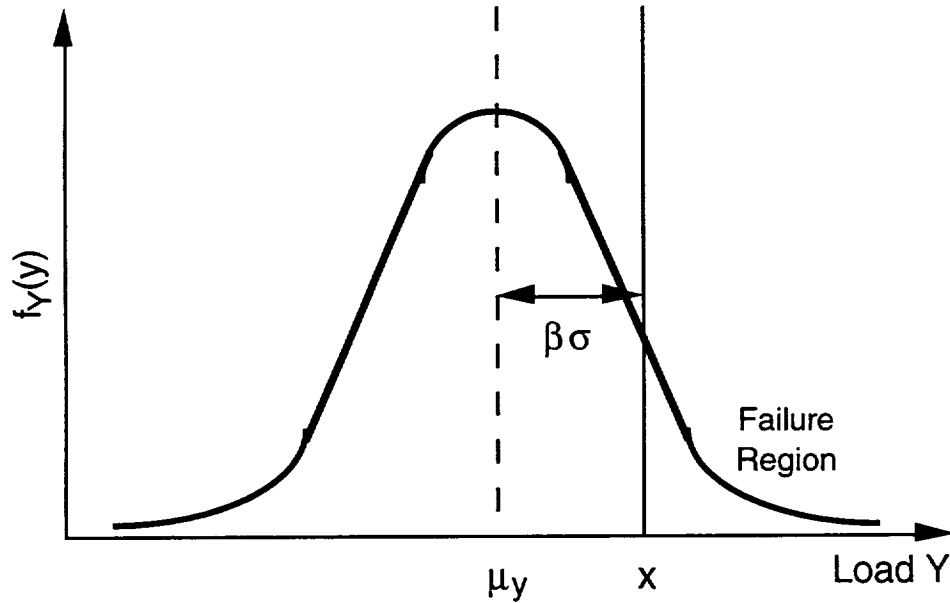


Figure 2-1: Reliability Analysis for One Variable

In the specific case of a Gaussian random variable the load  $Y$  can be “reduced” or transformed into a standard normal variable (SNV),  $y$ , with a mean of zero and a standard deviation of one unit:

$$Y = \mu_Y + y\sigma_Y \quad (2.1a)$$

or

$$y = (Y - \mu_Y) / \sigma_Y \quad (2.1b)$$

where  $\mu_Y$  is the mean value of  $Y$  and  $\sigma_Y$  is the standard deviation of  $Y$ . This variable can then be used in the previous reliability equation:

$$\text{Reliability} = P(x > \mu_Y + y\sigma_Y) \quad (2.2a)$$

or

$$\text{Reliability} = P((x - \mu_Y) / \sigma_Y > y) \quad (2.2b)$$

At this stage the variable on the right hand side of the inequality is a SNV and all the quantities on the left are known. These known quantities can then be reduced to a single value defined as the “reliability coefficient” or “reliability index,”  $\beta$ . The reliability of the system can then be stated as:

$$\text{Reliability} = P(\beta > y) \quad (2.3)$$

In the single variable case, this inequality defines a “safe region.” This region is the set of values of  $y$  for which the structure will not fail. The next step is to perform the integration implied by equation (2.3) over the appropriate region. For this case the reliability of the system is equal to the probability that a SNV is less than  $\beta$ . This is easily determined from tables of the SNV Distribution or through calculations of any other known distribution function. As before the probability of failure,  $P_f$ , is the complement of the reliability:

$$P_f = 1 - \text{Reliability} = 1 - P(\beta > y) \quad (2.4)$$

Note that the use of the words “reliability index” for  $\beta$  usually implies that the value  $x$  exceeds the mean of  $Y$  ( $\mu_y$ ),  $\beta > 0$ , implying a reliability greater than 0.5. The equations (2.3) and (2.4) are, however, valid in any circumstance.

This process is similar when the system involved has a random resistance (strength),  $X$ , in addition to a random applied load,  $Y$ . The probability of failure,  $P_f$ , of the system is still the probability that the load is greater than the resistance,  $P(Y > X)$ . To determine this probability, the density and distribution of the load,  $f_y(y)$  and  $F_y(y)$ , and of the resistance,  $f_x(x)$  and  $F_x(x)$ , must be determined. The distribution function  $F$  is defined by:

$$\begin{aligned} F_Y(y) &= P[Y \leq y] \\ &= \int_{-\infty}^y f_Y(z) dz \end{aligned}$$

Once these probability functions are known, an integration is performed over the unsafe region (where the load is greater than the strength) to determine the probability of failure. This is shown in figure 2.2:

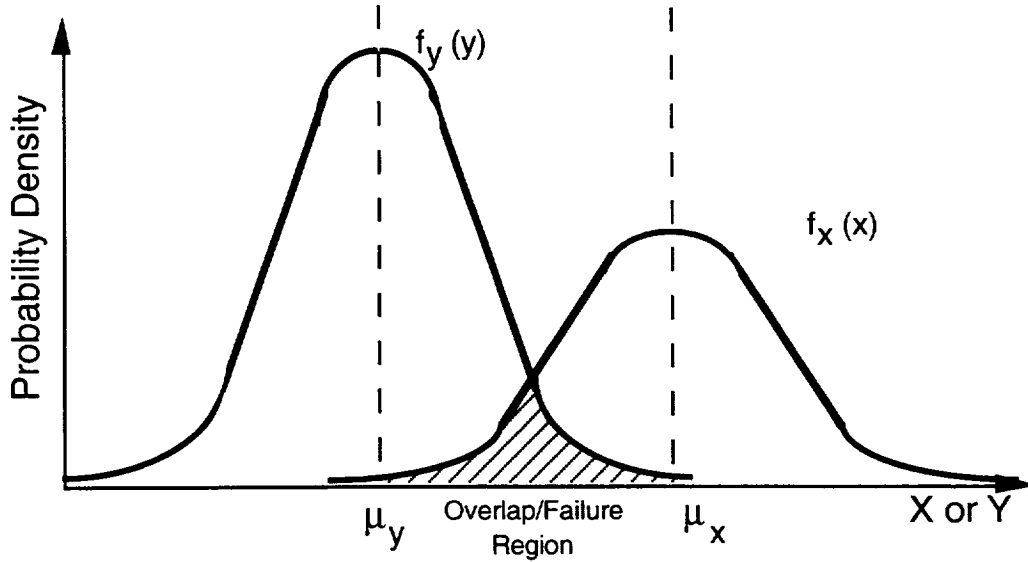


Figure 2-2: Reliability Analysis for Two Variable System

The required probabilities can be formulated in the following manner:

$$P_f = P(X < Y) = \sum_{\text{all } y} P(X < Y | Y = y) P(Y = y) \quad (2.5)$$

For continuous variables the summation becomes:

$$P_f = \int_0^{\infty} F_x(y) f_y(y) dy \quad (2.6a)$$

An alternative expression is given by:

$$P_f = \int_0^{\infty} [1 - F_y(x)] f_x(x) dx \quad (2.6b)$$

## 2.1 Design Variables

The two important characteristics that must be known about a design variable are how its values are distributed and if it is statistically independent of the other design variables. The variable distributions fall into two categories: normally and non-normally distributed. The majority of the techniques described in this report are for normal variables only. Non-normal variables can be used directly in the simple Monte Carlo method or approximated for use in the Rackwitz-Fiessler method; these special techniques will be briefly considered. All integration methods in this report are limited to uncorrelated variables. With known correlation factors between the variables, an orthogonal transformation can be performed on the variables such that the new variables are uncorrelated. This transformation requires the following steps for normally distributed variables. First the covariance matrix,  $[V]$ , is determined. The elements of the covariance matrix are defined as:

$$V_{ij} = E[(X_i - \mu_i)(X_j - \mu_j)] \quad (2.7)$$

$$E[\bullet] = \text{expected value of } [\bullet]$$

where the entries represent the dependence (correlation) of one design variable on the other. Since  $[V]$  is positive definite, there exists an orthogonal matrix,  $[T]$  such that  $[T][V][T]^T$  is a diagonal matrix. These diagonal elements are the eigenvalues of the covariance matrix. We can use this transformation matrix on the variables to produce uncorrelated variables by the following method:

$$\underline{Y} = T\underline{X} \quad (2.8)$$

With these new variables the covariance matrix is diagonal and therefore the variables  $\underline{Y}$  are uncorrelated. The simulation methods can also be performed with correlated variables if the correlation is known. In our application (composite materials) the material strength design variables may be correlated, but the nature of this correlation is unknown. We shall assume that the variables of strength and applied load are uncorrelated.

## 2.2 Determining The Most Probable Point of Failure

The general case consists of a design problem involving multiple random variables,  $\underline{X}$  (a vector of dimension  $n$ ). Note that  $\underline{X}$  may include both applied loads and strength (or resistance) values. To determine the probability of failure, a general criterion,  $g(\underline{X})$ , which is a function of the design variables, is required. This criterion defines a surface between the "safe region,"  $g(\underline{X}) < 0$ , and the "failure region,"  $g(\underline{X}) > 0$ , and is known as the "failure surface"  $g(\underline{X}) = 0$ . To facilitate calculation, the variables are converted into standard normal variables (SNV) by the following change of variables:

$$X_i = x_i \sigma_i + \mu_i \quad i=1 \text{ to } n \quad (2.9a)$$

or

$$x_i = (X_i - \mu_i) / \sigma_i \quad i=1 \text{ to } n \quad (2.9b)$$

where  $\mu_i$  is the mean value and  $\sigma_i$  is the standard deviation of the random variable  $X_i$ , the  $i$ th design variable. The failure surface is also transformed into a function of these reduced variables, giving  $g(\underline{x}) = 0$ . The variables,  $x_i$ , are normalized in units of standard deviation and measure the distance (in probability space) from the mean. The shortest distance from the origin (mean values) to the failure surface is then determined using a minimization technique. This distance is the reliability coefficient,  $\beta$ . The closest approach point on the failure surface,  $\underline{x}^*$ , is defined to be the most probable point (MPP) of failure since the density function for the SNV's decreases monotonically with distance from the origin. Using this information, the reliability can be calculated by employing a probability integration method which can provide a quick and accurate solution.

### 2.3 Importance Factors

Once the MPP has been determined, the importance of considering the variability (randomness) of an individual design variable can be determined. The importance is measured by the amount a variable's value at the MPP,  $x_i^*$ , deviates from its mean,  $\mu_i$ , in units of standard deviation. With reduced variables, this is simply the absolute value of the magnitude  $|x_i^*|$ . To compare the relative importance of the randomness in the individual design variables, "importance factors", which are simply the direction cosines, are defined:

$$I_i = (x_i^* / \beta) \quad i = 1 \text{ to } n \quad (2.10)$$

This ratio  $I_i$  of the variable's value at the MPP to the reliability coefficient is always between one and negative one. A variable whose randomness does not affect the design problem (has a low importance) will not vary from its mean value;  $x_i^* \approx 0$ . The more the importance factor deviates from this, the mean value, the greater the importance of the variability. These importance factors can be used to simplify a design problem. If a variable consistently has a factor showing low importance, then it may be possible to replace this random variable with its mean value in the failure criterion. This would simplify the problem by reducing its dimension. This is of minor value to the problems being analyzed in this report due to the small number (3 or 5) of variables being employed, but can be important in problems of larger dimension ( $n$ ).

### 2.4 Fast Probability Integration Techniques

Fast Probability Integration (FPI) methods use a variety of techniques to evaluate the measure of probability space enclosed or bounded by a failure surface. This is accomplished by first determining the most probable point (MPP) of failure in the reduced variable probability space. The reliability coefficient,  $\beta$ , is the distance from this point to the origin in units of standard deviation. For First Order methods, only this information is required. From the knowledge of  $\beta$  an approximation of the failure surface is made (a boundary shape is assumed) and the integration performed. For ease of calculation only simple shapes are assumed. In Second Order methods, additional information about the curvature of the failure surface at the MPP is required. These Second Order methods use both the MPP and the curvature to provide more accurate approximations.

#### 2.4.1 First Order Reliability Techniques

**Design Problems with Linear (Planar) Failure Surfaces.**—In some situations the failure surface is planar (linearly dependent on the design variables) and an exact solution can be calculated. In these situations the equation for the failure surface has the following form:

$$g(\underline{X}) = \alpha_1 X_1 + \alpha_2 X_2 \dots + \alpha_n X_n - \gamma \quad (2.11a)$$

or

$$g(\underline{X}) = \sum_{i=1}^n \alpha_i X_i - \gamma \quad (2.11b)$$

where  $\alpha_i$  are constant coefficients and  $\gamma$  is a constant. The equation of the failure surface after the normalization of variables becomes:

$$\sum_{i=1}^n \alpha_i \sigma_i x_i = \gamma - \sum_{i=1}^n \alpha_i \mu_i \quad (2.12)$$

and the distance to the MPP,  $\beta$ , can be determined directly from the geometry:

$$\beta = \frac{\sum_{i=1}^n \alpha_i \mu_i - \gamma}{\left[ \sum_{i=1}^n \alpha_i^2 \sigma_i^2 \right]^{1/2}} \quad (2.13)$$

For the applications being evaluated in this report a linear failure surface does not arise and is of minimal interest. In the more general case the same steps are performed, but a more robust technique is required.

**The Hasofer-Lind (H-L) Method.**—This is the simplest First Order, Second Moment reliability method used to evaluate probability. This technique approximates the failure surface (a hypersurface) by a hyperplane tangent to the failure surface at the MPP. This is shown in Figure 2-3 for a system of two design variables in SNV space:

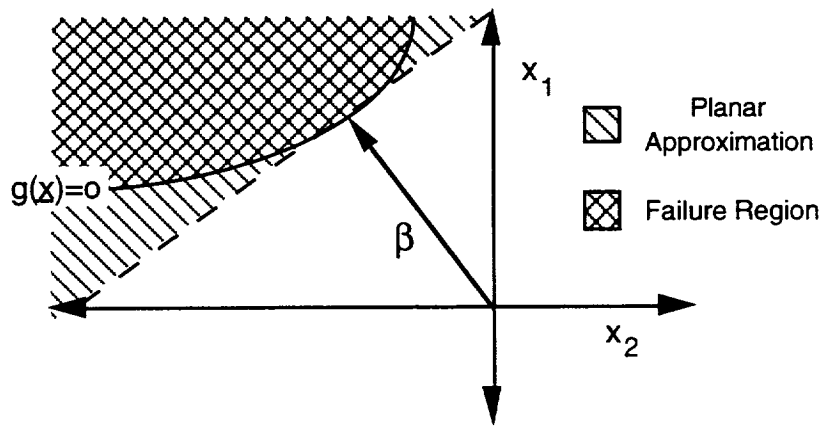


Figure 2-3: Hasofer-Lind FPI Technique

The plane can be defined by the first derivatives of the surface (at the MPP) and the MPP in terms of reduced variables:

$$\sum_{i=1}^n (x_i - x_i^*) (\partial g(\underline{x}) / \partial x_i)^* = 0 \quad (2.14)$$

First, however, the MPP point must be found. The usual analytical technique is to formulate a minimization problem with the distance to the MPP (in units of standard deviation) as the function to be minimized:

$$\begin{aligned} \text{Minimize:} \quad D(\underline{x}) &= \left[ \sum_{i=1}^n x_i^2 \right]^{1/2} \\ \text{Subject to:} \quad g(\underline{x}) &= 0 \end{aligned} \quad (2.15)$$

Note that the minimization problem 2.15 is formulated in terms of reduced variables. Once the problem is formulated a minimization routine will solve for the point that offers the smallest value of  $D(\underline{x})$ , satisfying  $g(\underline{x}) = 0$ . The minimization solution for  $D(\underline{x})$  gives the distance to the MPP,  $(\beta)$ , as well as the importance factors  $(I_i)$ . The tangent plane defined by 2.14 may now be found. However, the  $\beta$  value can be used directly to solve for the probability that the structure is within the safe region. By assuming the failure surface is planar and using reduced (SNV) variables, the value of the density function integrated over the hypervolume is found to be equal to the standard normal integral (distribution function) at  $\beta$ :

$$\text{Reliability} = \Phi(\beta) \quad (2.16)$$

while the probability of failure is the complement:

$$P_f = 1 - \Phi(\beta) = \Phi(-\beta) \quad (2.17)$$

where  $\Phi$  is the standard normal distribution function. The evaluation of  $\Phi$  either by computer or by comparison with tables will determine the reliability or the probability of failure. Note that equations (2.16) and (2.17), commensurate with figure 2-3, assume that reliability exceeds 50 percent; that is, that the origin is not part of the failure region. If this is not the case, a similar equation may be used.

**Hypersphere Centered at the Origin Approximation.**—The biggest potential problem with the H-L method is in the planar approximation. In the case of a failure surface that is convex about the MPP (shown in fig. 2-3), the surface is termed 'friendly' and the linear approximation is to some extent conservative. The alternative is a concave or 'dangerous' curvature. This results in a liberal approximation and may underestimate the actual probability of failure. This situation, for a system of two variables, is shown in figure 2-4:

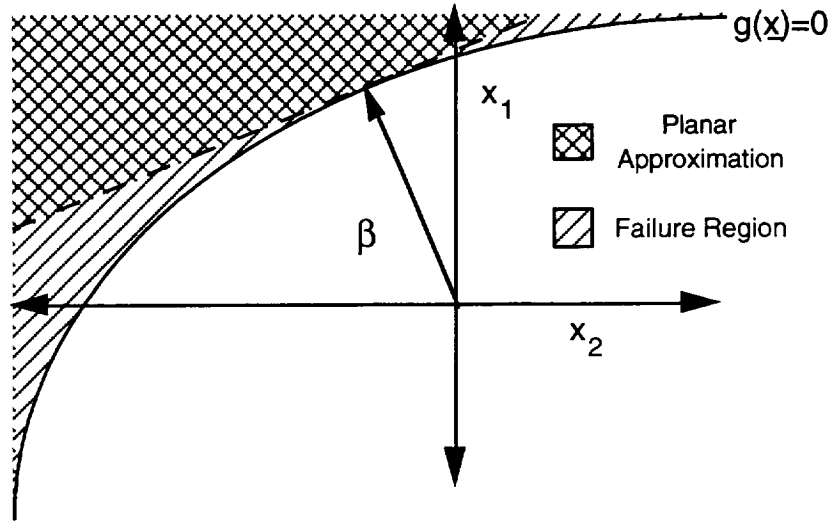


Figure 2-4: Hasofer-Lind with dangerous curvature

One response to this problem is to approximate the failure surface by a hypersphere centered at the origin. This method is also categorized as first order in that it is based on the distance to the MPP only. This approximation is shown in figure 2-5:

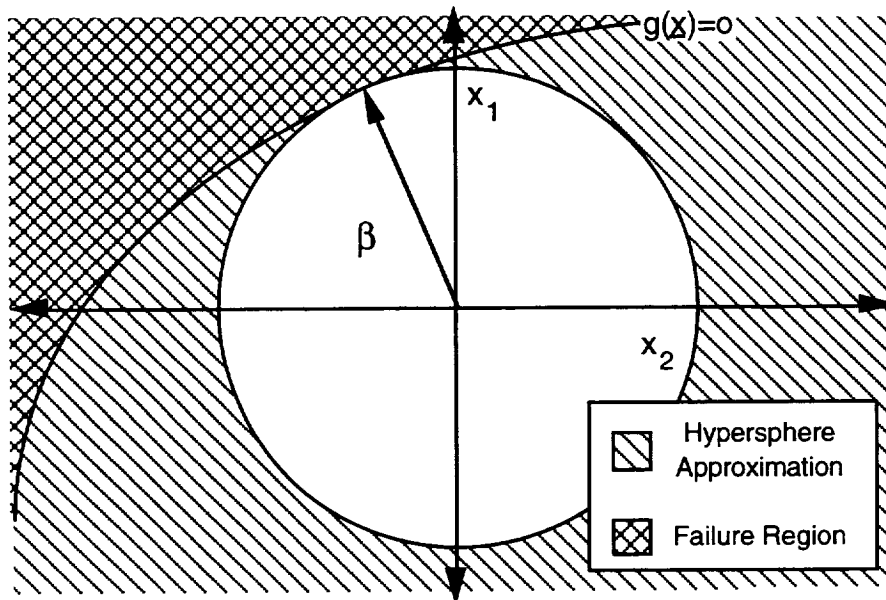


Figure 2-5: Origin Centered Hypersphere Approximation

The hypersphere is defined solely by the distance to the MPP,  $\beta$ :

$$\sum_{i=1}^n x_i^{*2} - \beta^2 = 0 \quad (2.18)$$

This approximation is extremely conservative in that it encloses the smallest possible volume in the safe region, based on the calculated value of  $\beta$ . By assuming the failure surface is spherical and by using SNV's, the sum of the squares ( $\sum x_i^2$ ) has a chi-squared distribution with  $n$  degrees of freedom. Therefore the value of the integration over the hypervolume can be obtained from the chi-squared distribution with  $n$  degrees of freedom:

$$\text{Reliability} = \chi_n^2(\beta^2) \quad (2.19)$$

while the probability of failure is the converse:

$$P_f = 1 - \chi_n^2(\beta^2) \quad (2.20)$$

This probability can be determined by computer or by the use of tables.

**Rackwitz-Fiessler Method.**—When the design variables are not normal variables, the standard normalization definition  $\{x = (X-\mu)/\sigma\}$  may still be used. There is no assurance, however, that the location of the closest point to the origin will be the most probable point. This difficulty arises from the fact that the decay in probability density may be unequal along the different design variable axes. If we can arrange for the probability description at the closest approach point to resemble (be locally equivalent to) a normal distribution, we may be able to force the closest point to be the MPP. The reliability would then be evaluated from the normal distribution. This is the rationale behind the Rackwitz-Fiessler method (ref. 12), which requires an iterative solution as outlined below.

Let the design variables  $X_i$  have any non-normal distribution. On the initial iteration, instead of normalizing the variables in the usual fashion, convert the minimization problem into a function of the original variables.

$$\text{Minimize:} \quad D(\underline{X}) = \left\{ \sum_{i=1}^n [(X_i - \mu_i)/\sigma_i]^2 \right\}^{1/2} \quad (2.21)$$

$$\text{Subject to:} \quad g(\underline{X}) = 0$$

where  $\mu_i$  and  $\sigma_i$  are the means and standard deviations of the non-normal variables (assumed known) and  $g(\underline{X})$  is the failure criterion in the original variables. This problem is mathematically equivalent to the minimization problem mentioned previously. The only difference is that the normalization of the variables is done within this minimization problem while in the previous problem the variables were normalized externally. The next step is to match the density and distribution functions of a similar normal distribution. A new distribution, with mean ( $\mu_i'$ ) and standard deviation ( $\sigma_i'$ ), is obtained for each variable,  $X_i$ , by matching the original distribution and density function at the closest approach point,  $\underline{X}^*$  (which is equivalent to  $\underline{x}^*$  in the normalized coordinate system), with those of a normal variable;

$$\Phi((X_i^* - \mu_i')/\sigma_i') = F(X_i^*) \quad (2.22)$$

$$\phi((X_i^* - \mu_i')/\sigma_i') = f(X_i^*) \quad (2.23)$$

where  $\Phi(\cdot)$  is the standard normal distribution function and  $\phi(\cdot)$  is the standard normal density function. To determine the new parameters (means and standard deviations) of the equivalent normal distributions equations (2.22) and (2.23) must be inverted to yield the following equalities.

$$\sigma_i' = \phi\{\Phi^{-1}[F(X_i^*)]\}/f(X_i^*) \quad (2.24)$$

$$\mu_i' = X_i^* - \sigma_i' \Phi^{-1}[F(X_i^*)] \quad (2.25)$$

The new values of the means and standard deviations found by equations (2.24) and (2.25) are inserted in the minimization problem 2.21, and the closest point  $\underline{X}^*$  is found. These two steps: solving the minimization problem and determining equivalent normal distributions, are repeated until a convergence criterion (either absolute or relative) is satisfied. The distance calculated is then used in the planar or hyperspherical approximation of the failure surface outlined above. From these approximations the reliability (or probability of failure) can be determined.

#### 2.4.2 Second Order Reliability Techniques

As in the first order methods, the second order methods approximate the hypervolume of the failure region by a simpler volume over which the probability integration may be more easily evaluated. This is equivalent to determining a simpler bounding surface which approximates the true surface. To improve the approximation of the failure surface beyond the level employed in the first order methods, additional information about the failure surface is required. In the second order methods the approximation relies on the curvature of the surface at the most probable point of failure in addition to the distance to that point. With this additional information, a more correct approximation surface can be used with confidence. The only limit to the shape chosen is the ability to match it to a known closed form expression for probability.

**The General Quadratic Form Solution.**—For implementation of any of the second order methods, the failure surface must be twice differentiable in the neighborhood of the checking point in the normalized coordinate system. Once the failure surface has been converted to the normalized coordinate system, the next step is to expand the failure surface,  $g(\underline{x}) = 0$ , into a second order Taylor series about the checking point,  $\underline{x}^*$  (ref. 5):

$$g(\underline{x}) = g(\underline{x}^*) + \sum_{i=1}^n \frac{\partial g(\underline{x})}{\partial x_i} \bigg|_{\underline{x}=\underline{x}^*} (x_i - x_i^*) + \frac{1}{2!} \left[ \sum_{i=1}^n \frac{\partial^2 g(\underline{x})}{\partial x_i^2} \bigg|_{\underline{x}=\underline{x}^*} (x_i - x_i^*)^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\partial^2 g(\underline{x})}{\partial x_i \partial x_j} \bigg|_{\underline{x}=\underline{x}^*} (x_i - x_i^*) (x_j - x_j^*) \right] = 0 \quad (2.26a)$$

or in matrix notation after rearrangements:

$$g(\underline{x}) = (\underline{x} - \underline{x}^*)^T \cdot [G_{\underline{x}}] \cdot (\underline{x} - \underline{x}^*) + 2 \cdot \{g_{\underline{x}}\}^T \cdot (\underline{x} - \underline{x}^*) + 2 \cdot g(\underline{x}^*) = 0 \quad (2.26b)$$

where  $[G_{\underline{x}}]$  is a matrix of second and mixed derivatives and  $\{g_{\underline{x}}\}$  is a vector of first derivatives with the following form:

$$[G_{\underline{x}}] = \begin{bmatrix} \frac{\partial^2 g(\underline{x})}{\partial x_1^2} & \frac{\partial^2 g(\underline{x})}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 g(\underline{x})}{\partial x_1 \partial x_n} \\ \frac{\partial^2 g(\underline{x})}{\partial x_2 \partial x_1} & \frac{\partial^2 g(\underline{x})}{\partial x_2^2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 g(\underline{x})}{\partial x_n \partial x_1} & \cdots & \cdots & \frac{\partial^2 g(\underline{x})}{\partial x_n^2} \end{bmatrix} \quad (2.27)$$

$$\{g_x\} = \left\{ \frac{\partial g(x)}{\partial x_1} \frac{\partial g(x)}{\partial x_2} \dots \frac{\partial g(x)}{\partial x_n} \right\}^T \quad (2.28)$$

By a process of linear transformations equation (2.26b) can be converted to one of two standard forms in a new coordinate system (defined as the z-system). These two possible transformed quadratic forms are defined by equations (2.29) and (2.30).

$$\sum_{i=1}^n \lambda_i (z_i - \delta_i)^2 = C_1 \quad (2.29)$$

$$\sum_{i=1}^m \lambda_i (z_i - \delta_i)^2 + \sum_{i=m+1}^n (c_i - z_i) = C_2 \quad (2.30)$$

where  $\lambda_i$  are eigenvalues of the  $[G_x]$  matrix, the  $\delta_i$  terms are the noncentralities in the z-system, and the constants  $C_1$ ,  $C_2$  and  $c_i$  ( $i=1,n$ ) are dependent on the linear transformation. The form these equations define is dependent upon the signs of the eigenvalue-to-constant ratios,  $\lambda_i/C_1$  and  $\lambda_i/C_2$ , for equations (2.29) and (2.30) respectively. If all ratios are greater than zero, the quadratic is positive definite. In the case of equation (2.29) this defines an ellipsoid centered at  $(\delta_1, \delta_2, \dots, \delta_n)$  with semi-axes  $[(C_1/\lambda_1)^{1/2}, (C_1/\lambda_2)^{1/2}, \dots, (C_1/\lambda_n)^{1/2}]$ . If any of the eigenvalues (or ratios) are zero, the equations represent cylindrical forms. In the case of eigenvalues with differing signs, the quadratic is indefinite and can have a variety of forms.

Because the variables in the z-system are SNV's, new variables can be defined

$$W = \sum_{i=1}^n \lambda_i (z_i - \delta_i)^2 \quad (2.31)$$

$$V = \sum_{i=1}^m \lambda_i (z_i - \delta_i)^2 + \sum_{i=m+1}^n (c_i - z_i) \quad (2.32)$$

where the variable  $W$  is a combination of noncentral chi-squared distributed variables only, while  $V$  contains a linear combination of normally distributed variables. The distribution function of  $W$  is the probabilistic content of a normal distribution where the spherical region is defined by equation (2.29). The distribution of variable  $V$  can also be defined in a similar fashion by equation (2.30), although the shape of the region is dependent upon the constants  $C_2$  and  $c_i$  ( $i=1,n$ ). From the definition of the safe domain (in either the initial x-system or in the transformed z-system) it can be shown that the probability of failure is estimated by

$$P_f = P(W > C_1) = 1 - F_W(C_1) \quad (2.33)$$

$$P_f = P(V > C_2) = 1 - F_V(C_2) \quad (2.34)$$

where  $F_W$  and  $F_V$  are the probability distribution functions of  $W$  and  $V$  respectively. A variety of expressions have been derived for use in estimating these density functions. These approximations require complicated integrations and as a result have limited applicability in the study of reliability. Two simpler forms for the quadratic approximation have probability distributions that are relatively easy to calculate and will be evaluated in detail in the following sections.

**Special Forms with Predetermined Principal Axes.**—In addition to their use in the exact curvature solution, the curvatures at the checking point or MPP  $\underline{x}^*$  can be used to approximate the surface with a predetermined shape. This results in a substantial savings in computation time but decreases the accuracy of

the results. The consequence of these approximations can only be evaluated when compared to the 'exact' solution.

The first step in performing this method is to determine the MPP of failure as previously described. The next step is to calculate the curvatures of the failure surface at this point, with an eye towards employing a single resultant curvature. This is accomplished by using the direction of the principal axis, which is a unit vector from the origin to the MPP. The coordinate system (x-system) is rotated in such a way as to allow the principal axis (nth axis) to lie along the position vector from the origin to the MPP. The most probable point is converted from  $(x_1^*, x_2^*, \dots, x_n^*)$ , in the old coordinate system, to  $(0, 0, \dots, \beta)$ , in the new coordinate system. This new system is defined as the y-system. This orthogonal rotation requires the transformation matrix,  $T$ , where:

$$\underline{X} = T\underline{Y} \quad (2.35)$$

This is shown in figure 2-6:

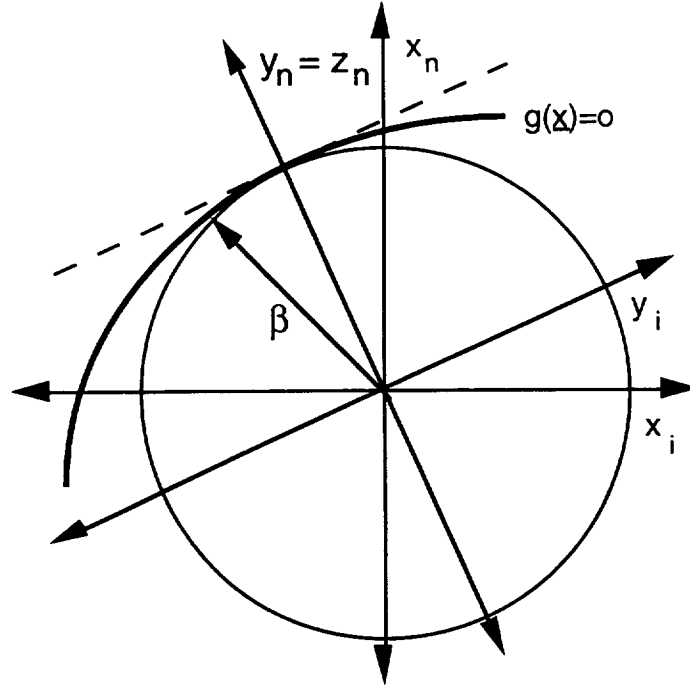


Figure 2-6: Rotated Coordinate Systems

Since we have all of the information about the failure surface in the unrotated coordinate system, the inverse of this transformation is of greater interest:

$$\underline{Y} = T^{-1}\underline{X} = T^T\underline{X} \quad (2.36)$$

The inverse and transpose are equal since  $T$  is an orthogonal matrix composed of unit vectors. To determine this matrix we use the MPP whose position is known in both coordinate systems. From this the nth row of the  $T^T$  matrix can be determined:

$$T_{ni}^T = x_i^*/\beta \quad (2.37)$$

Since the  $(n-1)$  secondary axes in the  $\underline{Y}$  coordinate system are arbitrary, but must be perpendicular to each other and lie in the hyperplane normal to the nth axis, the rows of the matrix have similar requirements.

The rows must represent unit vectors which are perpendicular to each other. The first step is to solve for row (n-1). The first n-1 positions (in the n-1 row) are assumed to be equal to one. Then, using the perpendicularity requirement, the n<sup>th</sup> element of row (n-1) is solved for by taking the dot product of row (n-1) with row (n) and setting the result to zero. This requires solving one linear equation with one unknown (element n of row (n-1)). The n-1 vector is then normalized. The computed vector is both a unit vector and perpendicular to the nth row. This process is then repeated for the next lowest row, (n-2). In this case the last two positions (n-1 and n) are non-arbitrary and must be solved for by taking dot products with the n-1 and nth rows. This requires solving two linear equations with two unknowns. These two steps are repeated for each row until the entire matrix is complete.

The second order and mixed derivatives at the MPP are calculated in the rotated coordinate system. This can be accomplished in two ways. The first is to use the matrix to transform the variables  $\underline{x}$  into the new coordinate system  $\underline{y}$ , then transform the failure criterion using these variables, and finally take the derivatives in the new system. The second method is to use the transformation matrix to determine the values of the derivatives in the rotated ( $\underline{y}$ ) coordinate system from those calculated in the unrotated ( $\underline{x}$ ) coordinate system. This requires the following calculations:

$$G_{y_i y_j} = \frac{\partial^2 g(\underline{y})}{\partial y_i \partial y_j} \Big|_{\underline{y}^*} = \frac{\partial^2 g(\underline{x})}{\partial x_p \partial x_q} \Big|_{\underline{x}^*} T_{i'p}^T T_{j'q}^T \quad (2.38)$$

$$g_{y_i} = \frac{\partial g(\underline{y})}{\partial y_i} \Big|_{\underline{y}^*} = \frac{\partial g(\underline{x})}{\partial x_p} \Big|_{\underline{x}^*} T_{i'p}^T \quad (2.39)$$

where  $[G_y]$  is the matrix of second and mixed derivatives,  $\{g_y\}$  is the vector of first derivatives in the rotated y-system and  $\underline{y}^* = T^T \underline{x}^*$ .

The coordinate system then undergoes an additional rotation with the  $y_n = z_n$  axis fixed, such that the mixed derivatives go to zero in the  $\underline{z}$  coordinate system. This is accomplished by performing an orthogonal transformation like the coordinate rotation described in Section 2.1. This final coordinate system is defined as the z-system, and its curvatures are the "principal" curvatures. The first and second derivatives ( $\{g_z\}$  and  $[G_z]$ ) are then calculated in the z-system using either of the two methods mentioned above. Using this matrix and vector, the curvature of the surface with respect to the variables in the final rotated coordinate system ( $\underline{z}$ ) can be calculated. For an n-dimensional surface there are (n-1) principal curvatures,  $K_i$ , calculated. This is accomplished by solving for the roots of the characteristic equation in the y-system:

$$\det \left( \frac{1}{g_{y_n}} [G_y] - K \cdot \mathbf{I} \right) = 0 \quad (2.40)$$

where  $g_{y_n}$  is the first derivative with respect to  $y_n$ , solution for  $K$  will give the vector of curvatures,  $[\mathbf{I}]$  is the (n-1) rank identity matrix and all derivatives are performed at the MPP. In the z-system the curvatures can be determined directly from the ratio of derivatives since the  $[G_z]$  matrix is diagonal:

$$K_i = \frac{G_{z_{ii}}}{g_{z_n}} \quad i=1 \text{ to } n-1 \quad \text{NO SUM} \quad (2.41)$$

These curvatures are identical to those calculated with equation (2.40). It is important to note that although use of the y- and z-systems simplify the calculation of the curvatures, these curvatures can not be identified with the original design variables due to the rotations performed. In both coordinate systems (y and z) a "dangerous" curvature (concave towards the origin) is defined by a negative second derivative while a "safe" curvature (concave away from the origin) is defined by a second derivative with a positive value. The first

derivative with respect to the primary (or nth) axis,  $g_{y_n}$ , is negative by definition, since it defines the boundary between the “safe” region (where the failure criterion is positive) and the “failure” region (where the failure criterion is negative). Taking these two conditions into account the sign of the curvature is opposite to the sign of the second derivative used to calculate it; a “dangerous” curvature is positive and a “safe” curvature is negative. The radius of curvature,  $R_i$ , is the reciprocal of the curvature:

$$R_i = 1/K_i \quad i = 1 \text{ to } n-1 \quad (2.42)$$

**Non-Central Hypersphere Approximation.**—The first geometric form to be utilized as an approximation is the non-central hypersphere. Unlike the hypersphere used in the first order method, this hypersphere is not centered at the origin. This is shown in figure 2-7:

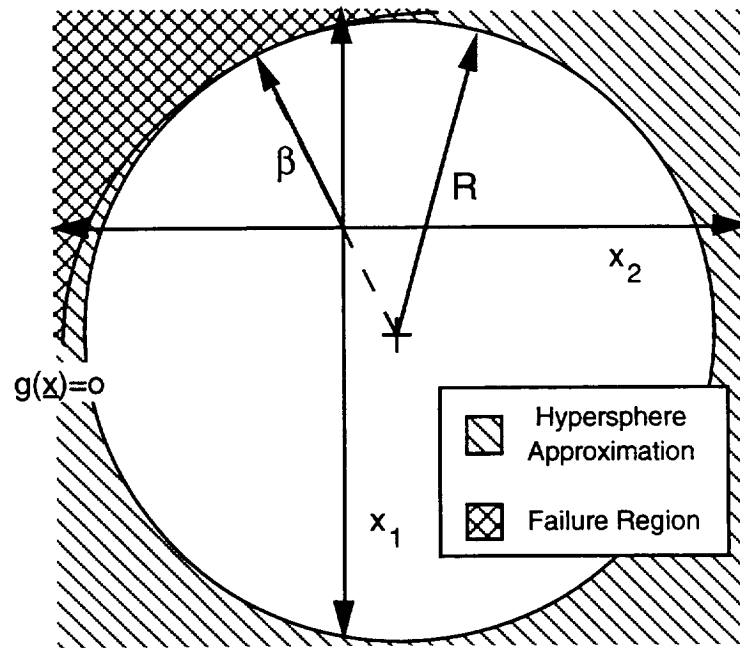


Figure 2-7: Noncentral Hypersphere Approximation

To determine the reliability one of the  $n-1$  radii of curvature values must be chosen, since a hypersphere only uses one radius. The choice of radius depends on the nature of the approximation desired. For this approximation only the positive and finite radii are considered; this eliminates any curvatures which are negative (“safe”) or zero (locally linear). If a conservative result is desired, the smallest radius, and thus the smallest hypersphere, is chosen. The opposite is true if a liberal result is desired. In addition to these choices, the average of the curvatures can be used. Once the radius is chosen, the equation of the hypersphere in the  $z$ -system becomes:

$$R^2 = [z_n - (\beta - R)]^2 + \sum_{i=1}^{n-1} z_i^2 \quad (2.43)$$

This hypersphere has radius  $R$  and center at the point  $(0, 0, \dots, \beta - R)$ , in the  $z$ -system. If the  $z$  variables are standard normal variables, the equation (2.43) is noncentrally chi-squared distributed with a noncentrality parameter  $\delta = [R - \beta]^2$ . This is used for reliability analysis:

$$\text{Reliability} = \chi^2_{n,\delta}(R^2) \quad (2.44)$$

and

$$P_f = 1 - \chi^2_{n,\delta}(R^2) \quad (2.45)$$

This distribution can be calculated from a known distribution equation (reference 11).

**Rotational Paraboloid Approximation.**—This is the other quadratic geometric form used in this report to approximate the hypervolume contained by the failure surface. This is shown in figure 2-8:

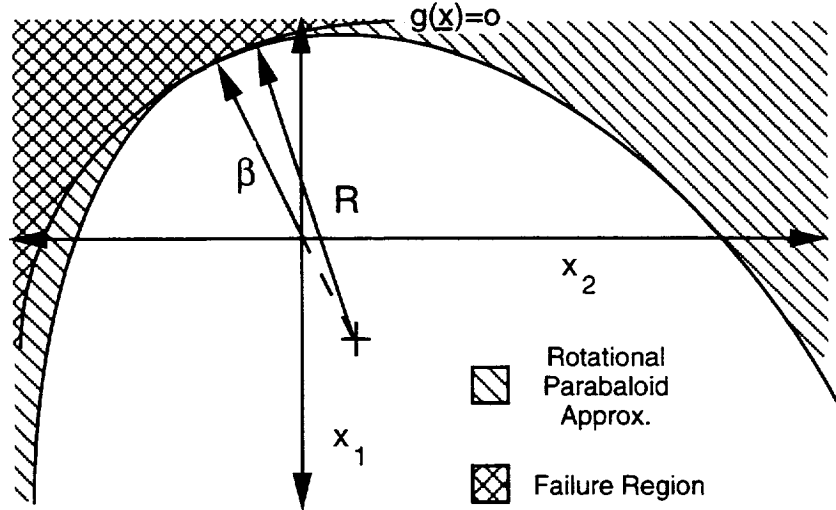


Figure 2-8: Rotational Paraboloid Approximation

The same choice of an appropriate radius of curvature must be made - either the minimum, maximum or average. As before, this choice is made from the set of positive and finite radii. Once the radius is chosen, the equation of the rotational paraboloid, in the  $z$ -system, becomes:

$$\frac{1}{2R} \sum_{i=1}^{n-1} z_i^2 + (z_n - \beta) = 0 \quad (2.46)$$

Since the  $z$  variables are SNV's, the first term in the equation is chi-squared distributed while the second term is normally distributed. Summing these two terms, the probability of failure becomes the probability of

a chi- squared variable convoluted with a noncentral normal variable's being greater than zero. This results in the following equation:

$$P_f = \int_0^{\infty} \Phi \left[ -\frac{1}{2R} t + \beta \right] f_{\chi^2_{n-1}}(t) dt \quad (2.47)$$

in which  $f_{\chi^2_v}$  is the density of a chi- squared variable with  $v$  degrees of freedom and  $\Phi$  is the SNV distribution function. In this case the density function has only  $n-1$  degrees of freedom due to the form of the previous equation. Evaluating the failure probability equation (2.47) requires numerical quadrature of the specified distribution and density functions.

The two presented quadratic approximations use equations based on those found in the article by Fiessler, Neumann, and Rackwitz (ref. 5). The specific form of the equations varies significantly from those referenced, however. To duplicate the results found in this report the modified equations (2.43), (2.46) & (2.47), as defined above, were required. The form of these equations in the article (ref. 5) are as follows:

$$R^2 = \sum_{i=1}^{n-1} z_i^2 + [z_n - (R + \beta)]^2 \quad (2.43')$$

$$\frac{1}{2R} \sum_{i=1}^{n-1} z_i^2 - (z_n - \beta) = 0 \quad (2.46')$$

$$P_f = \int_0^{\infty} \Phi \left[ -\frac{1}{2R} t - \beta \right] f_{\chi^2_{n-1}}(t) dt \quad (2.47')$$

In addition to these equations the noncentrality parameter was defined as  $\delta = [R + \beta]^2$ , and the center of the hypersphere was located at  $(0, 0, \dots, 0, R + \beta)$ . The majority of discrepancies regarded changes in sign. These differences may be caused by a difference in sign convention or in the definition of the normalized variables. All attempts to use these primed equations as stated in the two approximations (Non-Central Hypersphere and Rotational Paraboloid) gave incorrect results.

## 2.5 Monte Carlo Methods

**Simple Monte Carlo Technique.**—This is the simplest technique available to analyze a design problem and calculate  $P_f$ . It consists of sampling a number of random points, where the distribution of sample points is controlled by the distribution of the design variables. In each iteration, a value is generated for each design variable. These values are then tested in the failure criterion,  $g(\underline{x})$ . The results of this test are consistent with our previous definition:

- $g(\underline{x}) < 0$ : Point falls within failure domain
- $g(\underline{x}) = 0$ : Point falls on the failure surface
- $g(\underline{x}) > 0$ : Point falls within safe domain

This is repeated, with the results of each simulation being recorded. Once a sufficient number of iterations is completed, the probability of failure is calculated in the following manner:

$$P_f = \frac{\text{Number of Failure Simulations}}{\text{Total Number of Simulations}} = \frac{N_f}{N} \quad (2.48)$$

This is a very simple approach with only one serious drawback. As the reliability increases, the probability of failure decreases, and the total number of simulations required for an accurate estimate of  $P_f$  increases. For engineering design problems it is not unusual to have acceptable  $P_f$  values to be as low as  $1 \times 10^{-6}$  or  $1 \times 10^{-9}$ . For reasonable accuracy, a minimum of 20 to 30 sample failures are generally required, with a commensurate requirement on the total number of samples of say  $(20/P_f)$ . This is extremely costly in terms of computation time.

**Efficient Sampling Technique.**—By using a portion of the FPI methods outlined above, a more efficient sampling technique has been developed to address the inherent inefficiency of the basic Monte Carlo technique. In an attempt to reduce the number of samples, this technique is designed to generate variables in a limited sampling volume that follows the original distribution (within that limited sampling volume). As in the FPI technique outlined above, the first step is to determine the MPP and the associated  $\beta$ . Since this point is defined as the closest approach of the failure surface and the reduced variables are all SNV's, it follows that any point generated which lies at a distance (from the origin) less than  $\beta$  would fall within the safe domain. This n-dimensional sphere, of radius  $\beta$  and centered at the origin, termed the  $\beta$ -sphere, can then be considered a portion of the safe domain and removed from the sampling domain. Removal of this domain gives a substantial decrease in the number of samples required, since most points generated would otherwise fall near the origin within the  $\beta$ -sphere. This is shown in figure 2-9:

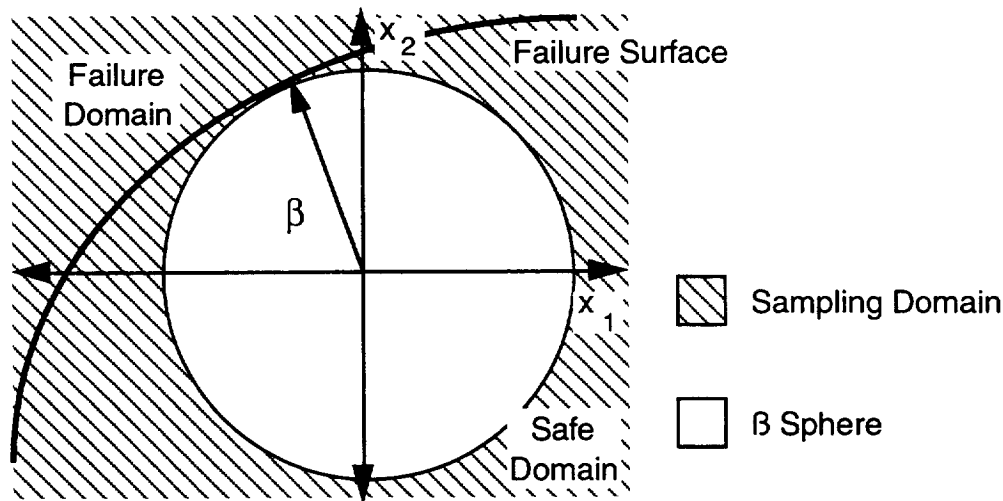


Figure 2-9: Limited Sampling Domain

The probability content contained within the  $\beta$ -sphere is calculated by using the formula for the cumulative chi-squared distribution with  $n$  degrees of freedom. By excluding the  $\beta$ -sphere from the sampling domain a substantial improvement in the efficiency is obtained.

To explain the steps used in this probability of failure estimator, the sampling domain must be split into the two portions defined by  $|\underline{x}| \leq \beta$  and  $|\underline{x}| > \beta$ . The probability of failure can then be shown in the following equation:

$$P_f = \Pr (g(\underline{x}) < 0) \quad (2.49)$$

Consider the two domains using the Theorem of Total Probability:

$$P_f = P\{g(\underline{x}) < 0 \mid |\underline{x}| \leq \beta\} \cdot P\{|\underline{x}| \leq \beta\} + P\{g(\underline{x}) < 0 \mid |\underline{x}| > \beta\} \cdot P\{|\underline{x}| > \beta\} \quad (2.50)$$

Instead of using the conditions based on  $|\underline{x}|$  and  $\beta$ , we can express them equivalently using  $|\underline{x}|^2$  and  $\beta^2$  for use with the chi-squared distribution:

$$P_f = P\{g(\underline{x}) < 0 \mid |\underline{x}|^2 \leq \beta^2\} \cdot P\{|\underline{x}|^2 \leq \beta^2\} + P\{g(\underline{x}) < 0 \mid |\underline{x}|^2 > \beta^2\} \cdot P\{|\underline{x}|^2 > \beta^2\} \quad (2.51)$$

Replacing the various probability terms with their known distributions and values results in:

$$P_f = 0 \cdot \chi_n^2(\beta^2) + P\{g(\underline{x}) < 0 \mid |\underline{x}|^2 > \beta^2\} \cdot (1 - \chi_n^2(\beta^2)) \quad (2.52)$$

In the first term the first part is replaced by zero due to lack of possible failures within the  $\beta$ -sphere while the second term is the probability the point is within  $\beta$ -sphere. In the second term the second part is the probability that a point lies outside of the  $\beta$ -sphere. The next step is to determine the truncated probability density function,  $f_{tr}(\underline{x})$ , given that  $|\underline{x}| > \beta$ :

$$f_{tr}(\underline{x}) = \begin{cases} K_{tr} \cdot f(\underline{x}) & \text{if } |\underline{x}| > \beta \\ 0 & \text{if } |\underline{x}| \leq \beta \end{cases} \quad (2.53)$$

$K_{tr}$  is a constant used to scale (normalize) the distribution, defined as

$$K_{tr} = 1/(1 - \chi_n^2(\beta^2)) \quad (2.54)$$

such that the integral of the density function over the applicable domain equals one:

$$\int_{|\underline{x}| > \beta}^{\infty} f_{tr}(\underline{x}) d\underline{x} = 1 \quad (2.55)$$

The final probability term can then be replaced by this truncated density function:

$$P\{g(\underline{x}) < 0 \mid |\underline{x}| > \beta\} = \int_{g(\underline{x}) < 0} f_{tr}(\underline{x}) d\underline{x} \quad (2.56)$$

$$= \int_{\text{all } \underline{x}} I(\underline{x}) \cdot f_{tr}(\underline{x}) d\underline{x} \quad (2.57)$$

where  $I(\underline{x})$  denotes the indicator variable defined as

$$I(\underline{x}) = \begin{cases} 1 & \text{if } \underline{x} \text{ is in the failure region} \\ 0 & \text{if } \underline{x} \text{ is in the safe region} \end{cases} \quad (2.58)$$

The probability of failure becomes:

$$P\{g(\underline{x}) < 0 \mid |\underline{x}| > \beta\} = E_{f_{tr}} [I(\underline{x})] \quad (2.59)$$

where  $E_{f_{tr}} [I(\underline{x})]$  denotes the expected value of  $I(\underline{x})$  with respect to the  $f_{tr}$  - distribution. By combining equations (2.52) and (2.59) the probability of failure is:

$$P_f = (1 - \chi_n^2(\beta^2)) E_{f_{tr}} [I(\underline{x})] \quad (2.60)$$

Like the simple Monte Carlo method, the estimated or inferred value of the indicator variable is:

$$E_{f_{tr}} [I(\underline{x})] \cong \frac{1}{N} \sum_{j=1}^N I(\underline{x}) = \frac{N_f}{N} \quad (2.61)$$

where  $N$  is the total number of simulations and  $N_f$  is the number of failure simulations. These simulations are generated according to the (truncated)  $f_{tr}$  - distribution. This is accomplished by converting the basic variables ( $X_1, X_2, \dots, X_n$ ) to polar coordinates ( $R, \Theta_1, \Theta_2, \dots, \Theta_{n-1}$ ) (ref. 9), where  $(\Theta_1, \Theta_2, \dots, \Theta_{n-1})$  defines the direction of  $\underline{X}$  while  $R$  defines the length of  $\underline{X}$  in  $x$ -space. If the  $X_i$ 's are standard normal variables it can be shown that the distance  $R$  is independent of the direction (defined by  $(\Theta_1, \Theta_2, \dots, \Theta_{n-1})$ ) (ref. 9 pg. 18.)

These simulations are realized by sampling a random direction unit vector  $\underline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_{n-1})$ . This requires the sampling of  $n-1$  dummy SNV's,  $Y_1, Y_2, \dots, Y_{n-1}$ , which are then normalized to a unit vector  $\xi$  in the following manner

$$\xi_i = \frac{Y_i}{|\underline{Y}|} \quad i=1 \text{ to } (n-1) \quad (2.62)$$

where  $|\underline{Y}|$  is the length of the vector formed by the  $n-1$  normal variables

$$|\underline{Y}| = \left\{ \sum_{i=1}^{n-1} Y_i^2 \right\}^{1/2} \quad (2.63)$$

This method is used to generate the  $(n-1)$  variables  $\theta_i$  which define the direction. The length variable,  $R$ , is generated following a truncated chi-squared distribution. This can be accomplished by using realizations of a chi-squared random variable and then discarding the values which do not fall within the reduced sampling domain ( $R > \beta$ ). Due to the rapid decline of the chi-squared density function (see figure 2-10A) a majority of the simulations would be discarded, which is highly inefficient. An alternate method proposed by Harbitz (ref. 6) involves the following variable transformation:

$$U = \exp \left( -\frac{1}{\alpha} R^2 \right) \quad (2.64)$$

where  $\alpha$  is a constant greater than two and dependent upon  $\beta$ . The sampling domain for  $U$ ,  $[u_1, u_2]$ , corresponding to the sampling domain for  $R$ ,  $[r_1, r_2]$ , where  $r_2 = (r_1 + 3)$  (beyond  $r_1 + 3$  the chi-squared density function is assumed relatively insignificant), is given by:

$$[u_1, u_2] = \left[ \exp\left(-\frac{1}{\alpha} r_2^2\right), \exp\left(-\frac{1}{\alpha} r_1^2\right) \right] \quad (2.65)$$

The density function for  $u$  is then proportional to  $g(u)$ , where:

$$g(u) = [-\ln(u)]^{(n/2)-1} u^{(\alpha/2)-1}, \quad (2.66)$$

This is a more uniform density function (see figure 2-10B) and therefore offers a more efficient sampling method.

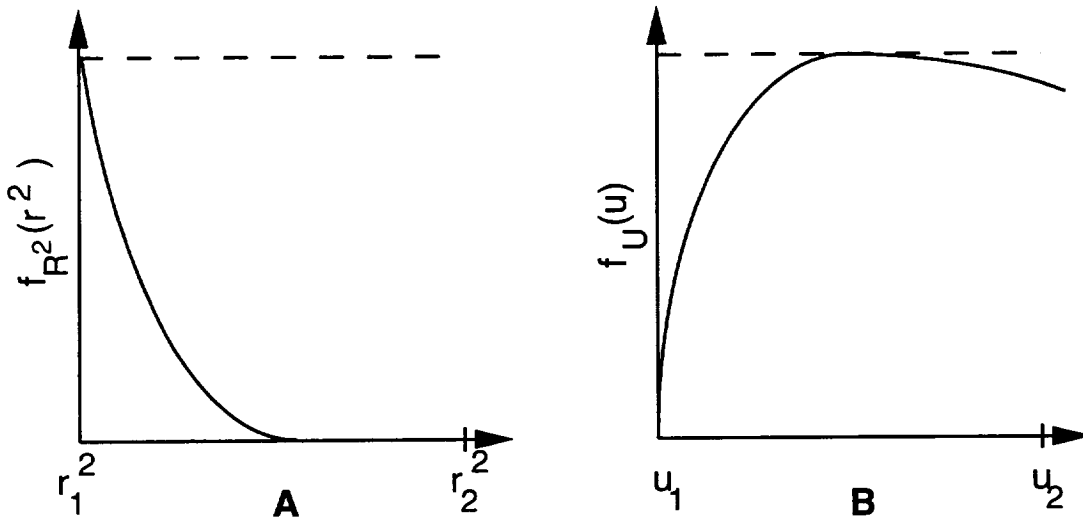


Figure 2-10: Sampling Density Function

In practice the following two procedures are considered for generating a variable which follows the distribution of equation (2.66) and shown in figure 2-10B:

Procedure 1: For large values of  $\beta$  or a small number of variables  $n$ , generate two stochastically independent rectangularly distributed (uniform density function) variables:

$$\begin{aligned} U &= \text{Rect} [u_1, u_2] \\ G &= \text{Rect} [0, g(u_{\max})] \end{aligned} \quad (2.67)$$

where

$$u_{\max} = \exp\left(-\frac{k-2}{\alpha-2}\right) \quad (2.68)$$

Sample random values of  $U$  and  $G$  until  $G < g(U)$ . This “acceptance sampling” is required to generate variables which follow the desired distribution. The corresponding  $R$ -value used in the simulation is then:

$$R = \sqrt{-\alpha \ln U} \quad (2.69)$$

This procedure is repeated until the desired number of simulations is generated based on the required number of sample failures.

Procedure 2: For small values of  $\beta$  and large  $n$  (number of variables), generate  $R$  by one of the following two equations depending upon if the number of variables is odd or even:

$$\text{For even } n: \quad R = -2 \ln \left\{ \prod_{i=1}^{n/2} V_i \right\} \quad (2.70)$$

$$\text{For odd } n: \quad R = -2 \ln \left\{ \prod_{i=1}^{(n-1)/2} V_i \right\} + X^2 \quad (2.71)$$

where  $V_i$  are independent unit rectangular variables and  $X$  is a standard normal random variable. This procedure is repeated until the desired number of simulations is generated.

If we now return to the reliability equation (2.60), a probability of failure estimate,  $P_f^*$ , is given by:

$$P_f^* = (1 - \chi_n^2(\beta^2)) \cdot \frac{N_f}{N} \quad (2.72)$$

When comparing this method to the simple Monte Carlo method, we see a substantial increase in efficiency. This can be shown by the following comparison of the total number of simulations required: for the simple Monte Carlo method  $N \approx 10/P_f$  (assuming that 10 failed simulations gives an accurate estimate) while for the Importance Sampling method  $N \approx 10 \cdot P_f / (1 - \chi_n^2(\beta^2))$ . This means that for a design problem with  $P_f = 1.0 \times 10^{-5}$  and  $\beta = 3.5$ ,  $N \approx 100$  for the Improved Sampling compared to  $N \approx 1 \times 10^6$  for the simple Monte Carlo method. In contrast to the FPI methods proposed earlier, this method does not require any prior knowledge of the shape of the failure surface.

### 3. RESULTS & DISCUSSION

There are three areas of reliability analysis being evaluated in this chapter. These will be covered in Sections 3.1, 3.2 and 3.3. As a standard multiaxial stress state, we shall consider the loading of an "off-axis" specimen with orientation angle,  $\Theta$ . The stress state is given by a reduction of equations (1.1), (1.2) and (1.3):

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} = \begin{Bmatrix} \cos^2 \Theta \\ \sin^2 \Theta \\ -\sin \Theta \cos \Theta \end{Bmatrix} \sigma_{xx} \quad (3.1)$$

The first portion of Section 3.1 will compare results obtained by using the techniques discussed in Chapter 2 for design problems with normally distributed strengths. Accuracy of the individual methods will be evaluated by comparing calculated reliability estimates. These comparisons will be performed over a range of orientation angles and applied stresses. In addition to accuracy, the other areas of comparison will include computational speed, intrinsic conservatism of the methods, and degree of difficulty in implementation. This comparison will be designed to demonstrate the methodology involved, and as such will not be exhaustive in its presentation. The second portion of Section 3.1 will evaluate the Rackwitz-Fiessler method. This is the basic method available to analyze systems with non-normally distributed design variables. This evaluation will include all areas of comparison discussed in the first portion. Section 3.2 will evaluate the sensitivity of

the probability of failure estimates to uncertainties in the original strength distribution parameters. The sensitivity study will be performed in a variety of cases with changes in parameters, orientation angles and applied stresses. Due to the difficulty in determining a closed form solution (with the notable exceptions of

laminae with  $0^\circ$  and  $90^\circ$  orientation angles) a numerical sensitivity study will be employed. As in the previous section, this analysis will be more concerned with demonstrating the methodology than in providing exhaustive detail. The final section (3.3) will provide and discuss the importance factors found during the reliability analysis.

**Confirmation of Published Solutions.**—To test the computational techniques used in performing these methods, a series of published design problems was duplicated. For the Hasofer-Lind method, the problem duplicated was analyzing composite failure using Hashin's Criterion with the applied stresses as the random variables [ref. 2]. These results of the published analyses gave relatively low reliability (on the order of 75%) and were not compared to results calculated with other methods. The results found from our analysis did not match the published results. Consultation with the principal author (Elishakoff) eventually led to the determination of the errors in these results. These errors were caused by a change of units which took place between the original problem [ref. 3] and the journal article form. The original problem was posed in units of MSI which were converted in the published problem to GPa (with a conversion factor of 6.895). This conversion was not performed correctly on the standard deviation values (in [ref. 2]: standard deviation of all stress distributions = 1 GPa, correct value: 6.895 GPa). Once this problem was corrected, the published results and our calculated results matched nearly exactly.

Additional analysis of other design problems were also duplicated (refs. 1, 4 and 5). For the Rackwitz-Fiessler method the problem duplicated involved flexure of a steel beam; see Ang & Tang (ref. 1). The three random variables had lognormal or asymptotic extreme-value distributions. Our analysis of this design problem offered results similar to the published results; however, more iterations were required. This is likely due to Ang & Tang's use of an analysis technique which was tailored to design problems having lognormal and asymptotic distributed variables, while our approach allowed for analysis of design problems containing variables with any nonnormal distribution.

The second order methods (both the Noncentral Hypersphere and Rotational Paraboloid) were tested using analysis of a bar under a series of tensile loads (ref. 5). This design problem was analyzed by all of the FPI methods and the results were compared with "exact" values calculated with the Monte Carlo method. When this design problem was analyzed using second order methods, extremely accurate estimates were possible. This was attributed to two reasons. Because this was only a two dimensional design problem (two variables), there was only a single radius of curvature involved; thus the radius used in the analysis was the exact radius at the MPP. Also the radius of curvature was of the same order as the reliability coefficient. This resulted in a failure surface which had a relatively extreme dangerous curvature and limited the surface to shapes which are close to the approximations. The Importance Sampling Method was tested using a design problem with a failure surface which, in reduced variables, was a rotational paraboloid (ref. 6). This example offered both calculated probabilities of failure and information on the efficiency of the method. The probability of failure results analyzed with our computational techniques were close to the published results. The deviations between the two sets of results are likely due to the small number of simulations used ( $N = 100$ ). The portion of this example involving efficiency did not match well with the published results, however. It was determined that the matrix of  $\alpha$ -values found in table 2 of reference 6 were transposed (i.e., the  $\alpha$ -values found in the first row of the table actually belong in the first column). Analysis of these problems using the corrected  $\alpha$ -values offered efficiency results which matched those published.

### 3.1 Comparison of Techniques

**Accuracy.**—The details of the comparison of lamina strength results using various techniques can be seen in tables 3-1 to 3-3. Each table consists of the reliability analysis of a lamina with a specific orientation under a series of axially applied stresses. At each stress level the reliability is analyzed by the basic Monte Carlo and Efficient Sampling Techniques, as well as four FPI methods: the Hasofer-Lind Method (Planar Approximation), the Central Hypersphere Approximation, the Non-Central Hypersphere Approximation and the Rotational Paraboloid Approximation. These results are compared to the reliability estimate calculated by the Monte Carlo method, which is believed to be the most accurate, provided sufficient trials can be realized. The number of simulations performed by the Monte Carlo method is chosen to give an extremely accurate (defined "exact") estimate. This is done by requiring the number of simulated failures to be larger than 50; this requires  $50/P_f$  simulations.

The material used in these simulations is a unidirectional fiber, polymer matrix composite (PMC) consisting of carbon (graphite) fibers in an epoxy matrix. The nominal or mean values for the five strength properties based on typical values are as follows:

Longitudinal Tensile Strength: 1400 MPa  
 Longitudinal Compressive Strength: 1200 MPa  
 Transverse Tensile Strength: 70 MPa  
 Transverse Compressive Strength: 140 MPa  
 In-Plane Shear Strength: 70 MPa

The distribution of these properties will be assumed to follow a normal distribution with a coefficient of variation (standard deviation/mean) of 0.10. For the second order approximations (the Non Central Hypersphere and the Rotational Paraboloid Approximations) a series of radii are calculated. The infinite radii are eliminated. The second order methods chosen are incapable of analyzing safe curvatures ( $K_i < 0$ ) so these are also discarded. From the remaining curvatures the largest curvature (smallest radius of curvature) is chosen. Typically there exists only one positive and finite (defined "dangerous") curvature for cases analyzed by the the Tsai-Hill and Tsai-Wu failure criteria. One zero value curvature ( $R = \infty$ ) was usually discarded for Tsai-Hill cases, while the set of discarded curvatures for Tsai-Wu cases usually consisted of one zero value and two "safe" curvatures ( $K_i < 0$ ).

Table 3-1: Reliability Analysis of a 30° Gr/Ep Composite

Failure Criterion:	Tsai - Hill			Tsai - Wu		
Applied Stress (MPa)	70	90	120	70	90	120
MPP-Normalized ( $\chi^*$ )						
SLt	-0.0010	-0.0016	0.0002	-0.0565	-0.0705	-0.0403
SLc	—	—	—	0.0690	0.0909	0.0578
STt	-0.173	-0.310	-0.363	-0.455	-0.630	-0.390
STc	—	—	—	0.140	0.165	0.0906
SLTs	-5.52	-4.09	-1.70	-5.28	-3.69	-1.12
Reliability Index $\beta$	5.52	4.10	1.74	5.30	3.74	1.19
Radius of Curvature R	1190	570	225	440	235	115
Probability of Failure: Ctl. Hypersph.:	108.e-8	75.8e-5	38.9e-2	3.52e-5	1.55e-2	9.23e-1
Planar Approximation	1.67e-8	2.02e-5	4.13e-2	5.83e-8	9.04e-5	1.17e-1
Non Central Hypersphere	1.78e-8	2.29e-5	4.96e-2	7.57e-8	13.7e-5	1.63e-1
Rotational Paraboloid	1.67e-8	2.02e-5	4.13e-2	5.97e-8	9.35e-5	1.21e-1
"Exact" Value	—	2.02e-5	4.13e-2	—	9.73e-5	1.33e-1
Importance Sampling	1.61e-8	2.10e-5	4.72e-2	5.59e-8	9.77e-5	1.27e-1

From table 3-1 it can be seen that the various FPI methods, with the exception of the central hypersphere, predict approximately the same probability of failure. This is due to the large radii of curvature found in all cases, which force all the models to behave like a linear (planar) approximation. It is only at the highest stress levels, where the radius of curvature is at its lowest value, that these predictions differ significantly. In all cases the central hypersphere approximation differs substantially from the "exact" value and thus is of interest only insofar as it offers a lower limit on reliability. In all six of the cases shown in

table 3-1 (three loads with two criteria) only one curvature (per case) was positive and finite or in reliability terms “dangerous”. This curvature was chosen for the second order evaluation. As mentioned previously, by choosing the largest curvature (and thus the smallest radius), the approximation becomes conservative. Including this curvature appears to have a slight effect on the Tsai-Wu analysis while the Tsai-Hill analysis is virtually unchanged. In the Tsai-Hill analysis only one curvature is ignored; using the Tsai-Wu analysis, three curvatures are ignored since they are either zero or negative. These extra curvatures can describe a surface that is significantly different than the one defined by the single “dangerous” curvature, and caution must be used. The results calculated by the Importance Sampling method do not depend upon curvature and thus are unaffected by this concern. The Non-Central Hypersphere approximation like the central hypersphere, tends to be conservative, but is much more accurate. Due to the flatness of the surface at the MPP, the probability of failure calculated by the two second order methods and the probability found from the planar approximation are approximately equal.

Table 3-2: Reliability Analysis of a 45° Gr/Ep Composite

Failure Criterion:	Tsai - Hill			Tsai - Wu		
Applied Stress (MPa)	55	70	85	55	70	85
MPP-Normalized ( $\bar{x}^*$ )						
SLt	0.0067	0.0016	0.0004	-0.0379	-0.0451	-0.0197
SLc	—	—	—	0.0727	0.0739	0.0446
STt	-0.540	-2.92	-1.43	-5.36	-2.05	-0.909
STc	—	—	—	0.0835	0.202	0.104
SLTs	-5.68	-2.93	-1.40	-1.07	-2.75	-0.953
Reliability Index $\beta$	5.71	4.14	2.00	5.47	3.44	1.32
Radius of Curvature R	523	529	360	540	310	215
Probability of Failure:						
Ctl. Hypersph:	3.97e-7	65.6e-5	262e-3	2.93e-5	374e-4	88.3e-2
Planar approximation	5.75e-9	1.72e-5	2.28e-3	4.73e-8	2.94e-4	9.31e-2
Non Central Hypersphere	7.28e-9	2.73e-2	4.95e-2	7.09e-8	4.01e-2	19.2e-2
Rotational Paraboloid	5.75e-9	2.26e-4	4.54e-2	2.29e-8	3.01e-4	9.46e-2
“Exact” Value	—	5.13e-5	3.03e-2	—	4.27e-4	10.6e-2
Importance sampling	1.29e-8	6.21e-5	3.52e-2	5.01e-8	3.14e-4	9.81e-2

In table 3-2 the results calculated by the various FPI methods (except the Central Hypersphere) are extremely similar. Only the failure probability values calculated by the Planar Approximation are systematically below the “exact” values. This is expected because this approximation is liberal in dangerous curvature cases. All other reliability methods (second order FPI and Importance sampling) appear to have equal accuracy. This is significant in the first two simulations performed using the Tsai-Hill Failure Criterion (Applied Axial Stress of 55 and 70 MPa respectively) in that the radius chosen for the approximation was the smaller of the two dangerous curvatures calculated. Therefore even a conservative second order approximation may result in a relatively accurate reliability analysis. It is important to note the change in the nature of the curvature as a function of the applied stress level using the Tsai-Hill failure Criterion. This shows that shape of the failure surface can change based solely on the stress applied. Because assumptions about this shape may be used when determining which reliability method should be chosen, this property must be considered.

Table 3-3: Reliability Analysis of a 60° Gr/Ep Composite

Failure Criterion:	Tsai - Hill			Tsai - Wu		
Applied Stress (MPa)	55	60	65	55	60	65
Normalized MPP ( $\bar{x}^*$ )						
SLt	0.00052	0.00057	0.00058	-0.0141	-0.0134	-0.0115
SLc	—	—	—	0.0273	0.0263	0.0231
STt	-3.70	-3.03	-2.33	-3.49	-2.77	-2.02
STc	—	—	—	0.0588	0.0657	0.0649
SLTs	-0.342	-0.385	-0.396	-0.53	-0.561	-0.523
Reliability Index $\beta$	3.72	3.06	2.37	3.53	2.82	2.09
Radius of Curvature R	1100	909	766	907	755	640
Probability of Failure:						
Ctl. Hypersph:	3.13e-3	2.51e-2	1.33e-1	2.82e-2	1.58e-1	4.99e-1
Planar Approximation	9.96e-5	1.12e-3	8.99e-3	2.05e-4	2.38e-3	1.84e-2
Non Central Hypersphere	11.4e-5	1.32e-3	10.7e-3	3.20e-4	3.87e-3	2.95e-2
Rotational Parabaloid	11.4e-5	1.32e-3	10.7e-3	3.20e-4	3.85e-3	2.93e-2
"Exact" Value	10.9e-5	1.25e-3	10.9e-3	2.31e-4	2.75e-3	2.28e-2
Importance Sampling	12.0e-5	1.43e-3	11.8e-3	1.16e-4	1.42e-3	1.24e-2

It can be seen from table 3-3 that the nature of the curvature is strongly dependent upon the orientation angle for the Tsai-Hill failure criterion. It has been found by a variety of simulations that any Tsai-Hill reliability analysis performed with an angle of orientation greater than 50° will result in all curvatures being safe. This is apparent in table 3-3. In this type of case the probability of failure determined by the planar approximation would be conservative. The only method capable of providing a potentially more accurate result is the the Importance Sampling method. However, in the three simulations of this type performed the Importance Sampling method did not offer an improvement in the accuracy of the calculated results over FPI. Nearly all of the demonstrated reliability methods performed equally well when the Tsai-Wu failure criterion was used.

The one type of situation in which the Importance Sampling Technique does not perform well is that of low reliability ( $\beta$  approaches zero). As  $\beta$  decreases the volume contained within the  $\beta$ -sphere also decreases and the efficiency of this method approaches that of the basic Monte Carlo method. This may be of little importance in application of this method to actual design problems in that low reliability cases will be rarely modelled.

Two additional orientations where these techniques can be compared is 0° and 90°. Unlike the previous cases, these orientations can result in design problems with closed form solutions. This is due to simplifications that these orientations cause in the failure criteria used. At these orientations the Tsai-Hill criterion reduces to:

$$\text{For } 0^\circ: \quad \frac{\sigma_{xx}^2}{S_{Lt}^2} - \frac{\sigma_{xx} \sigma_{yy}}{S_{Lt}^2} + \frac{\sigma_{xy}^2}{S_{Tt}^2} = 1 \quad (3.2a)$$

$$\text{For } 90^\circ: \quad \frac{\sigma_{yy}^2}{S_{Lc}^2} - \frac{\sigma_{yy} \sigma_{xx}}{S_{Lc}^2} + \frac{\sigma_{xy}^2}{S_{Tt}^2} = 1 \quad (3.2b)$$

If we further assume that the composite is under a single load applied in the X direction, this reduces to:

$$\text{For } 0^\circ: \quad S_{Lt} = \sigma_{xx} \quad (3.3a)$$

$$\text{For } 90^\circ: \quad S_{Tt} = \sigma_{xx} \quad (3.3b)$$

These two equalities offer closed form solutions. When this case is analyzed by the methods given above, the only one which accurately predicts the reliability is the Hasofer-Lind method, although this type of analysis is unnecessary. With this same assumption, the Tsai-Wu failure criterion reduces to

$$\text{For } 0^\circ: \quad S_{Lt} - S_{Lc} + \sigma_{xx} = \frac{S_{Lt} S_{Lc}}{\sigma_{xx}} \quad (3.4a)$$

$$\text{For } 90^\circ: \quad S_{Tt} - S_{Tc} + \sigma_{xx} = \frac{S_{Tt} S_{Tc}}{\sigma_{xx}} \quad (3.4b)$$

Even though these equations are not as simple as those derived from the Tsai-Hill failure criterion, when they are used the reliability estimates calculated by the Hasofer-Lind method are still quite accurate (within 5% of the "exact" value).

**Conservatism.**—All FPI methods being evaluated in this report may suffer from a predetermined conservatism or liberalism depending upon the nature of the design problem being analyzed. In each case the decision of which method to use must be made. This decision will determine if the calculated probability of failure will be liberal (potentially too low an estimate) or conservative (usually higher than necessary). In all cases the Central Hypersphere approximation offers a conservative estimate but at the same time is usually highly inaccurate (see tables 3-1 to 3-3). The planar approximation on the other hand can be either liberal or conservative depending on the curvature of the surface at the MPP. When a failure surface has all safe curvatures then this approximation will be conservative; while if even one curvature is dangerous this approximation may be liberal. The second order approximations are even more complicated due to the variety of possible curvatures used to approximate the surface shape. Using a large curvature (small radius) is conservative, while a small curvature (large radius) is liberal. These second order methods appear to offer the closest approximation to the actual surface however.

**Computational Speed.**—The speed of the methods is difficult to describe due to their case dependency. Computational speed is also dependent upon the initial guess used to determine the MPP. Because all methods, with the exception of the basic Monte Carlo method, require the determination of  $\beta$ , this may be ignored. The two important items which affect the speed of the FPI methods are the order of the approximation and the difficulty in evaluating the mathematical expression used to approximate the probability of failure. Once  $\beta$  has been determined, the first order methods require the evaluation of the normal distribution function for the planar approximation and the chi-square distribution function for the central hypersphere. Both of these functions are relatively simple to evaluate computationally.

Additionally the second order methods require the determination of a rotation matrix, the evaluation of first and second derivatives at the MPP, and the determination of the eigenvalues of the characteristic equation to determine the curvatures. These steps add substantially to the computational time. The two mathematical functions we must evaluate for these methods are more complicated than those used for the first order. The Non-Central Hypersphere approximation requires the evaluation of the non-central chi-squared distribution function. This is done by integrating the chi-squared density function using a numerical quadrature where each density function value is represented by the summation of an "infinite" number of terms. Accurate results for this calculation are only possible for failure surfaces (g-functions) with small radii at the checking point; for large radii, generation of accurate results are not possible using available computational techniques. For the Rotational Paraboloid the calculation of the distribution function requires

the integration by numerical quadrature of the convolution of a normally distributed term and a chi-squared term. These two methods are far more computationally time consuming than either of the first order methods.

The Monte Carlo methods available are by far the slowest computationally, due to their repetitive nature. This can be shown in the following example:

**Example 3-1.**—As mentioned in the Introduction, the choice of the strengths as design variables is based on a knowledge of composite manufacturing techniques, while the choice of the stresses as deterministic values is relatively arbitrary. It is not difficult to imagine a design problem in which the applied stresses are also random variables. In this situation the problem would have six random variables if analyzed with the Tsai-Hill failure criterion and eight if analyzed with Tsai-Wu. Analysis of this type of problem with the FPI methods given would not require substantially more computational time. The only change would be the difficulty of the search for the MPP. The speed of the Monte Carlo methods on the other hand would be greatly affected. Because the number of random variables that are required during each simulation is directly related to the amount of computational time required to perform each simulation, this change would increase the amount of time required to perform the basic Monte Carlo proportionately. The Importance Sampling drastically cuts down the number of simulations required but at the same time requires the determination of  $\beta$  as in the FPI methods. It is still far quicker than the simple Monte Carlo in nearly all cases.

**Implementation Difficulty.**—The simplest method to program is the basic Monte Carlo method. All that is required is random number generation to obtain realizations of the strengths (and stresses, if required), followed by an acceptability test using the failure criterion. The first order methods and the Importance sampling are only slightly more difficult. These require using a minimization routine with the failure criterion as the equality constraint. Once this is performed and the reliability coefficient,  $\beta$ , is known, the next step is either the analysis of a distribution function for the first order methods, or a series of random numbers generated in the reduced domain for Importance Sampling. These two methods are still relatively simple to program if a packaged minimization routine is available. By far the most difficult methods to implement are the second order methods. This is due in part to the number of steps involved in each method and the complexity of the individual steps. Once the reliability coefficient is known the coordinate system must be rotated to align one of the new axes with the vector to the MPP. This is followed by calculation of the first and second-order derivatives which are used to determine the curvatures. An eigenvalue problem is solved for the principal curvatures. These principal curvatures are examined and a single positive curvature is chosen to determine the radius of curvature based upon the conservatism desired. The parameters  $\beta$  and  $R$  are used to evaluate the distribution functions which govern the particular approximation. Another factor that affects the implementation difficulty is the number of program changes required to perform each case. As the methods get more complicated more information must be known about the problem definition. The only case-dependent requirement of the first order FPI and Monte Carlo methods is the failure criterion equation. The Importance Sampling method requires this criterion and a parameter for the truncated distribution used to generate random numbers. The second order FPI methods require the equations of the first and second derivatives of the failure criterion at the MPP in addition to the failure criterion equation for each case. These additional steps must be examined in the context of the computational speed and accuracy desired.

**Non-Normal Variables.**—A brief comparison of methods can also be made for design problems involving independent variables with non-normal distributions. The accuracy of the probability of failure estimates determined by the individual methods will be evaluated over a range of orientation angles and applied stresses. Computational speed, intrinsic conservatism, and the difficulty of implementing the methods will also be compared briefly. In this comparison two FPI methods will be examined; the Rackwitz-Fiessler method with a planar approximation and the Rackwitz-Fiessler method with the Central hypersphere approximation. These results will be compared to the reliability estimates calculated by the Hasofer-Lind method using normal distributions that are "equivalent" (possess the same means and standard deviations) to the original non-normal distributions and to the basic Monte Carlo method, which is believed to be the most accurate. The number of simulations performed by the Monte Carlo method will be chosen to give a very accurate (defined "exact") estimate. As in the normal variable section this is done by requiring the number of failures calculated (in each simulation) to be larger than 50.

The material used in this analysis is a ceramic matrix composite (CMC). The specific material chosen consists of SiC/RBSN. This CMC is evaluated by examining its three strength properties with nominal (average) values:

Longitudinal Ultimate Strength: 680 MPa  
Transverse Strength: 27 MPa  
In-Plane Shear Strength: 53 MPa

These properties are all for tensile loading of the material (the shear strength is independent of the loading sign). From experiments, the distribution of these properties has been shown to follow a Weibull distribution with the nominal parameters:

	Shape Parameter	Location Parameter
Longitudinal Ultimate Strength:	5.2	741 MPa
Transverse Strength:	10.9	28 MPa
In-Plane Shear Strength:	7.5	56 MPa

Due to the limited amount of experimental data for compressive strength, this composite will be evaluated by the Tsai-Hill failure criterion only. The results of the accuracy analysis can be seen in the following table.

Table 3-4: Tsai-Hill Reliability Analysis of CMC Composite

Orient. Angle	30°		45°		60°	
Applied Stress	40 MPa	50 MPa	20 MPa	30 MPa	12 MPa	20 MPa
MPP: $S_{Lt}$	579 MPa	625 MPa	653 MPa	653 MPa	653 MPa	653 MPa
$S_{Tc}$	17.8 MPa	25.7 MPa	10.2 MPa	15.7 MPa	9.04 MPa	15.2 MPa
$S_{LTs}$	21.0 MPa	24.8 MPa	52.6 MPa	51.9 MPa	53.0 MPa	52.8 MPa
Reliability Coefficient: $\beta$	4.09	2.87	4.16	2.93	4.45	3.03
Probability of Failure: Ctl. Hypersph.:	8.02e-4	4.08e-2	6.06e-4	3.54e-2	1.88e-4	2.73e-2
Planar Approximation	2.55e-5	2.02e-3	1.58e-5	1.70e-3	4.33e-6	1.24e-3
"Exact" Value	4.67e-4	2.54e-3	4.64e-4	9.29e-3	1.92e-4	9.28e-2
"Equivalent" $\beta$	3.44	2.83	4.01	2.68	4.28	2.79
"Equivalent" Hasofer-Lind	4.85e-5	2.31e-3	3.10e-5	3.72e-3	9.25e-6	2.64e-3

From table 3-4 it can be seen that neither of the two FPI methods available show systematic accuracy. The only pattern in the inaccuracy of the methods is that the planar approximation always gives probability of failure ( $P_f$ ) results which are smaller than the "exact" value (under-estimated) while  $P_f$  calculated with the Central Hypersphere is always higher (over-estimated). In terms of application to actual design problems these calculated  $P_f$  estimates would only be valuable as limits: the central hypersphere approximation being the lower limit and the planar approximation the upper. The values in the final two rows of the table are calculated by the Hasofer-Lind method using normal distributions "equivalent" (having the same means and standard deviations) to the original Weibull distributions. In all cases this relatively unsophisticated approach, with a planar approximation, calculates probability of failure values that are closer to the "exact" value than those calculated with Rackwitz-Fiessler method. However, these estimates are still too liberal for general application. It appears from the cases examined that as the load increases (probability of failure increases) the planar approximation becomes more accurate. To test this hypothesis three additional cases were tested: an orientation angle of 30° with an applied axial load of 60 MPa, an orientation angle of 45° with an applied axial load of 40 MPa and an orientation angle of 60° with an applied axial load of 60 MPa. In each case the probability of failure estimated using the Rackwitz-Fiessler method with a planar approximation was about 65% of the "exact" value. This is slightly more accurate than the previous cases. Why these results appear to follow this pattern is unknown, however.

These two approximations used in the Rackwitz-Fiessler method require nearly identical amounts of computational time. Both require the same iterative approach to calculate the reliability coefficient. The only difference is the evaluation of the two distribution functions. The chi-squared distribution function required for the Central Hypersphere is slightly more difficult to evaluate than the normal distribution function used in the planar approximation. The difference is minimal, however. These approximations have the same intrinsic conservatism when applied to the Rackwitz-Fiessler method as the dangerous curvature case does with the other FPI methods. The planar approximation is liberal while the central hypersphere is conservative. Computational techniques used to evaluate these two approximations are also equally difficult to implement.

### 3.2 Sensitivity

**Mean Strengths.**—Any variation in the mean value of the strength distribution can have a strong effect on the probability of failure of the system. The rate that this variation changes  $P_f$  gives the sensitivity of the probability of failure to the mean value. To analyze this sensitivity a series of reliability estimates are made by varying the mean value while all other properties (orientation angle, applied load and other distribution parameters) are held constant. The calculated failure probabilities are then compared to determine the relationship between  $P_f$  and the mean strength values. From observation it can be determined that the log of  $P_f$  is roughly linearly dependent upon the mean value of strength. This is to be expected in that all of the distribution functions used in the FPI methods are to some extent exponential in form. To show how this type of numerical sensitivity study could be performed the following example is provided:

**Example 3-2.**—In this example we determine the sensitivity of the probability of failure to the mean value of the shear strength. Preliminary results have shown that varying the mean shear strength results in the greatest change in the probability of failure (as compared to the axial or transverse strengths). The lamina being analyzed is a graphite fiber/epoxy matrix lamina with a  $45^\circ$  orientation angle. The coefficient of variation is 0.10. The other parameters for the strength distributions are taken to be the nominal values listed in Section 3.1. The applied stress is 90 MPa.

This case will use the Tsai-Hill failure criterion. From the previous section, it has been shown that the Hasofer-Lind method gives accurate results for design problems of this type and thus will be used for this analysis.

A linear plot does not present a simple relationship. If the data are viewed on a semi-log plot (log scale for the probability of failure) then an exponential relationship can be seen; see figure 3.1.

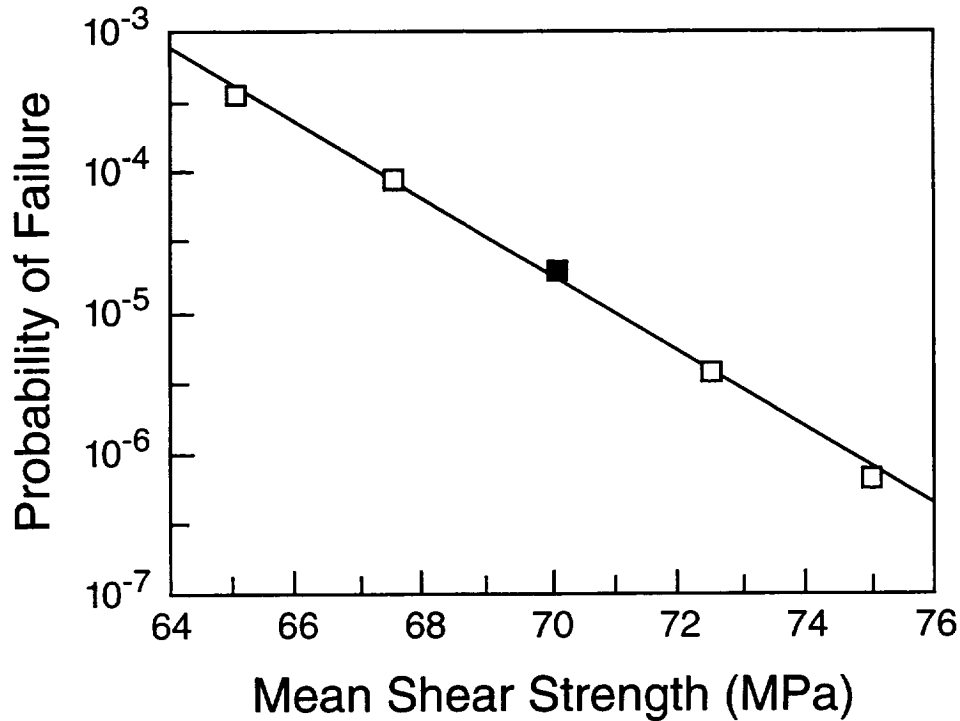


Figure 3-1: Sensitivity to Shear Strength Mean Value (Tsai-Hill failure criterion)

To analyze this case a linear relationship is assumed between the log of  $P_f$  and the mean shear strength value. The equation of the line determined by a least squares method is

$$\text{Log}(P_f) = 13.99 - 0.2678 \mu_{S_{LTs}} \quad (3.5a)$$

or

$$P_f = 10^{(13.99 - 0.2678 \mu_{S_{LTs}})} \quad (3.5b)$$

where  $\mu_{S_{LTs}}$  is the mean value of the shear strength distribution in MPa. With this equation a relatively accurate prediction of the probability of failure can be made at any mean strength value as long as it remains close to the values used to generate equation (3.5b).

The sensitivity  $P_f$  to changes in the mean strength value can be evaluated using the first derivative of equation (3.5b):

$$\frac{dP_f}{d\mu_{S_{LTs}}} = 0.616 \times 10^{(13.99 - 0.2678 \mu_{S_{LTs}})} \quad (3.6)$$

**Standard Deviation of Strengths.**—The probability of failure is also sensitive to changes in the standard deviations of the strength distributions. To analyze this sensitivity, a series of  $P_f$  calculations are made while changing the value of the standard deviation with all other properties (orientation angle, applied load and other distribution parameters) kept constant. The failure probabilities are calculated using a reliability method and then compared. From this comparison, a numerical relationship between the probabilities of failure and the standard deviation is determined. A semi-log plot is again expected to yield the most insight into the results. The following example is provided to show how this type of numerical sensitivity study would be performed.

**Example 3-3.**—In this example, we determine the sensitivity of the probability of failure to the standard deviation value of the shear strength distribution. The lamina being analyzed is of graphite fiber/epoxy matrix with a 30° orientation angle. The coefficient of variation for all unchanging distributions is 0.10. The means of the strength distributions are the nominal values listed in Section 3.1. The applied load is 90 MPa.

This example uses the Tsai-Hill failure criterion. From Section 3.1 it has been shown that Importance Sampling is the only method which gives accurate results for this type of design problem. The relationship between the standard deviation and probability of failure can be seen in figure 3-2.

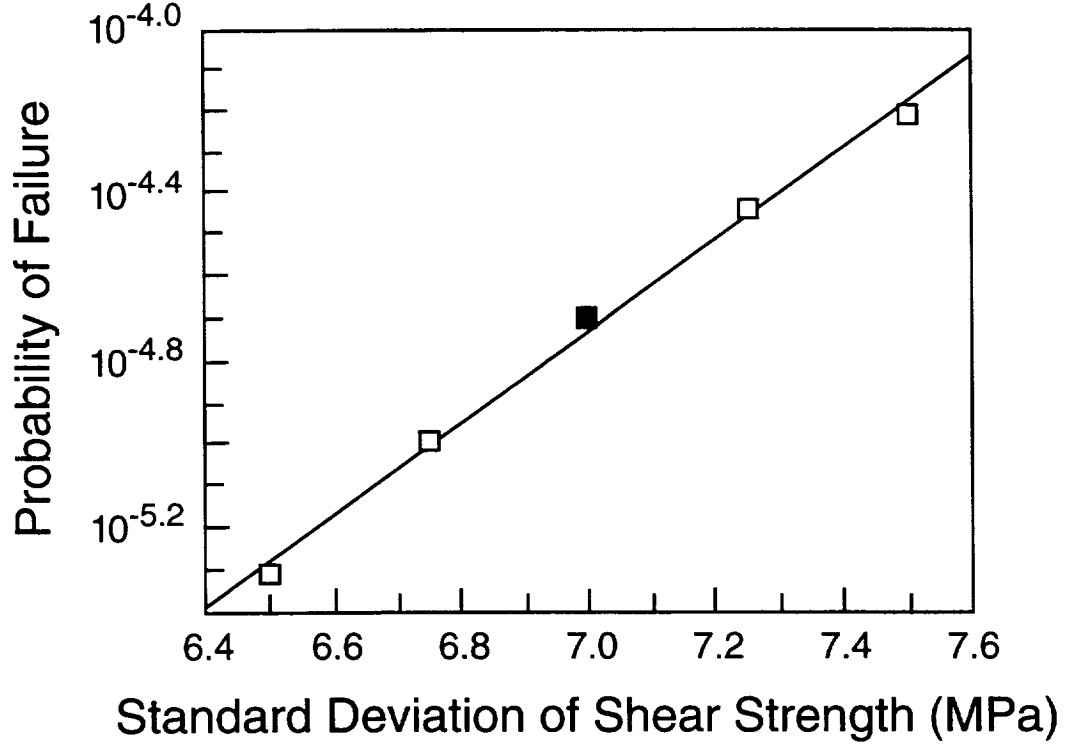


Figure 3.2: Sensitivity to Shear Strength Standard Deviation.

Although these values do not lie quite as smoothly on a line as in figure 3-1, a linear relationship between the log of the probability and the standard deviations of transverse strength can still be found by a least squares approach. This relationship is:

$$\text{Log } (P_f) = -12.45 + 1.104 \sigma_{SLTs} \quad (3.9a)$$

or:

$$P_f = 10^{(-12.45 + 1.104 \sigma_{SLTs})} \quad (3.9b)$$

where  $\sigma_{SLTs}$  is the standard deviation of the shear strength distribution in MPa. This gives relatively accurate predictions of the probability of failure in the neighborhood of the nominal values. The sensitivity of the probability of failure is the first derivative with respect to the standard deviation. For the shear strength the sensitivity is

$$\frac{dP_f}{d\sigma_{SLTs}} = 2.539 \times 10^{(-12.45 + 1.104 \sigma_{SLTs})} \quad (3.10)$$

**Applied Stress.**—The probability of failure is also sensitive to changes in the deterministic input of the design problem. In this report the two variables that have been considered deterministic are the applied stress and the orientation angle. We expect the variability in the orientation angle to be quite small, since the lay-up process is usually accurate. On the other hand, the applied stress may have uncertainty associated with it. To analyze the sensitivity of the probability of failure to the applied load, a series of  $P_f$  estimates are made by changing the value of the applied stress with all other properties (orientation angle and strength distributions) held constant. The calculated failure probabilities are then compared. A numerical relationship between the probabilities of failure and the applied stress is determined from this comparison. A semi-log plot is chosen for clarity of representation. The following example is provided to show how this type of study would be performed.

**Example 3-4.**—In this example, we will determine the sensitivity of the probability of failure to the axially applied stress,  $\sigma_{xx}$ . The lamina being analyzed is a graphite fiber/epoxy matrix with a  $30^\circ$  orientation angle. The coefficient of variation for all distributions is 0.10. The mean values of the strength distributions are the nominal values listed in Section 3.1. The nominal value of applied stress is 90 MPa.

This example uses the Tsai-Hill failure criterion. The second order method using the Rotational Paraboloid approximation gives accurate results for this type of design problem and will be used for this analysis. The relationship between the applied stress and probability of failure can be seen in figure 3-3. Typical of brittle materials, there is a great sensitivity to the applied stress.

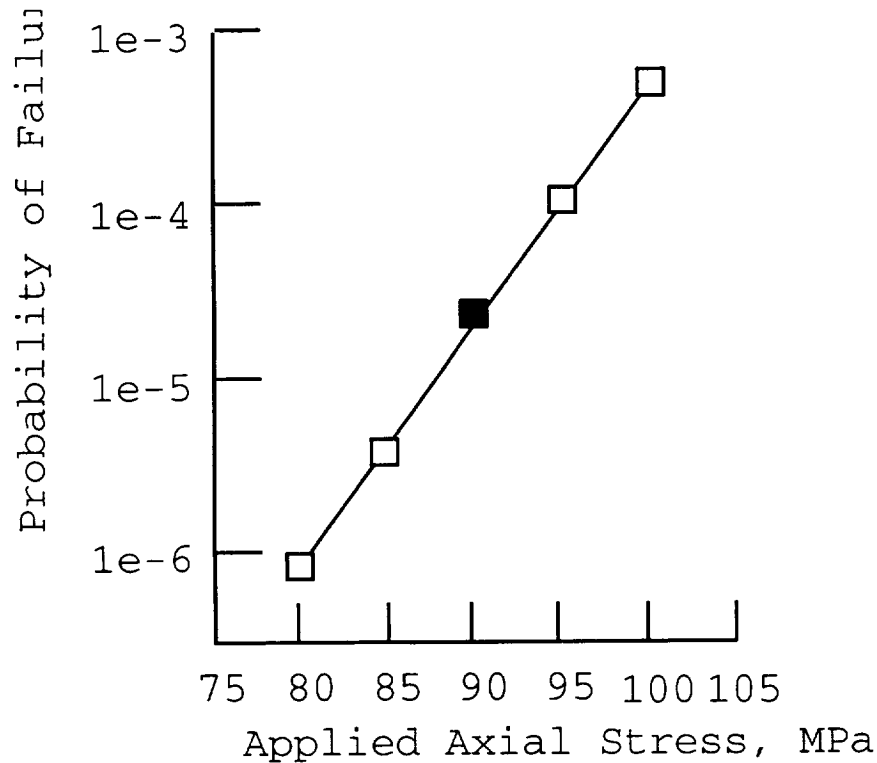


Figure 3-3: Sensitivity to Applied Load.

This relationship is analyzed by the same approach as in the previous examples with the following results

$$\text{Log}(P_f) = 0.145 \sigma_{xx} - 17.68 \quad (3.12)$$

$$P_f = 10^{(0.145\sigma_{xx} - 17.68)} \quad (3.13)$$

where  $\sigma_{xx}$  is the applied axial load in MPa.

### 3.3 Importance Factors

Importance factors are a measure of the impact a specific variable's randomness has on the overall design problem. They are also the direction cosines of the MPP position vector in reduced variable space. The direction cosine associated with a variable is the ratio of the reduced variable value at the MPP to the distance to the MPP, which distance is the reliability coefficient  $\beta$ . The importance factors are defined as follows

- $I_{Lt}$  = Axial Tensile Strength Importance Factor
- $I_{Lc}$  = Axial Compressive Strength Importance Factor
- $I_{Tt}$  = Transverse Tensile Strength Importance Factor
- $I_{Tc}$  = Transverse Compressive Strength Importance Factor
- $I_{LTs}$  = In-plane Shear Strength Importance Factor

The results of the importance factor analysis are shown in tables 3-5 to 3-7. Each table consists of the results calculated from reliability analysis of a specific lamina with a given orientation angle, using both the Tsai-Hill and Tsai-Wu failure criteria. The strength parameters for the graphite/epoxy composite analyzed are the same as those used in section 3.1. The stress values applied coincide with those used to generate the probability of failure values found in tables 3-1 to 3-3. The importance factors are found for the three strength distributions used for the Tsai-Hill failure criterion and the five used for the Tsai-Wu failure criterion.

Table 3-5: Importance Factors for a 30° Gr/Ep Composite

Failure Criterion:	Tsai - Hill			Tsai - Wu		
Applied Stress, MPa	70	90	120	70	90	120
$I_{Lt}$	$1.8 \times 10^{-4}$	$3.9 \times 10^{-4}$	$1.1 \times 10^{-4}$	$1.1 \times 10^{-2}$	$1.9 \times 10^{-2}$	$3.4 \times 10^{-2}$
( $I_{Lc}$ )	—	—	—	$1.3 \times 10^{-2}$	$2.4 \times 10^{-2}$	$4.9 \times 10^{-2}$
$I_{Tt}$	$3.1 \times 10^{-2}$	$7.6 \times 10^{-2}$	$21 \times 10^{-2}$	$8.6 \times 10^{-2}$	$17 \times 10^{-2}$	$33 \times 10^{-2}$
( $I_{Tc}$ )	—	—	—	$2.6 \times 10^{-2}$	$4.4 \times 10^{-2}$	$7.6 \times 10^{-2}$
$I_{LTs}$	$\approx 1.0$	$\approx 1.0$	0.977	0.996	0.987	0.941

From Table 3.5 it can be seen that the randomness of the shear strength dominates all of the other strength variabilities for this lamina. This domination is nearly complete in the two cases analyzed with the Tsai-Hill failure criterion, with applied loads of 70 and 90 MPa. In contrast the variability of the axial strengths (both tensile and compressive) have nearly no effect on the results. This lack of importance can also be seen in the transverse tensile strength when using either failure criterion. The variability of the transverse compressive strength on the other hand has a moderate effect (in Tsai-Wu analyzed problems only) on the design problem. The meaning of these importance factors can be seen in following examples:

**Example 3-5a.**—In the last case (Tsai-Wu failure criterion with applied axial load of 120 MPa) there are a range of importance factors. We can compare the effect of these factors by replacing a single variable with its mean value in the failure criterion and reducing the number of variables in the design problem by one. In this example, the axial tensile strength  $S_{Lt}$  is replaced by its mean value (1400 MPa). The reliability coefficient calculated with the variable is 1.19 while the coefficient with the deterministic value is still 1.19. This shows that the randomness of a variable with a importance factor of 0.034 is insignificant to the design problem.

**Example 3-5b.**—We continue to use the last case but in this example replace the transverse tensile strength variable, which has an importance factor of 0.33, by its mean value (70 MPa). The reliability coefficient calculated with the variable is 1.19 but the coefficient with the deterministic value is now 1.25. This is only a 5% change in the reliability coefficient but results in a –11% change in the probability of failure estimated with the Hasofer-Lind method. This shows the significance of the randomness of a variable with a factor slightly larger than say 0.1.

**Example 3-5c.**—In this example, using the previous results, we replace the shear strength variable, which has an importance factor of 0.941, by its mean value (70 MPa). The reliability coefficient calculated with the deterministic value is 54.1. This is 45 fold change in the reliability coefficient and results in a probability of failure which cannot be easily estimated with the Hasofer-Lind method due to computer limitations. This shows the overwhelming dominance of the randomness of a variable whose importance factor is above say 0.9.

The lack of importance of the axial strength's variability is expected due to the dependence of the failure criteria upon the reciprocal of the strengths squared. The relatively large nominal values of the axial strength (when compared with other strengths) reduces their impact on the failure criteria. The lack of importance of the transverse strength randomness may be due to the small orientation angle and loading scheme used.

Table 3-6: Importance Factors for a 45° Gr/Ep Composite

Failure Criterion:	Tsai - Hill			Tsai - Wu		
Applied Stress, MPa	55	70	85	55	70	85
$I_{Lt}$	$1.2 \times 10^{-3}$	$3.9 \times 10^{-4}$	$2.0 \times 10^{-4}$	$6.9 \times 10^{-3}$	$1.3 \times 10^{-2}$	$1.5 \times 10^{-2}$
$(I_{Lc})$	—	—	—	$13 \times 10^{-3}$	$2.1 \times 10^{-2}$	$3.4 \times 10^{-2}$
$I_{Tt}$	$95 \times 10^{-3}$	0.705	0.715	0.980	0.596	0.689
$(I_{Tc})$	—	—	—	$15 \times 10^{-3}$	$5.9 \times 10^{-2}$	$7.9 \times 10^{-2}$
$I_{LTs}$	0.995	0.708	0.700	0.196	0.799	0.722

Shear strength variability continues to dominate the cases found in table 3-6. Additionally the randomness of the transverse tensile strength does have a strong effect on the cases analyzed with the Tsai-Wu failure criterion. This overall increase in importance (when compared with the results from table 3.5) is due to the larger orientation angle. The variabilities of the remaining three strengths (axial tensile, axial compressive and transverse compressive) have very slight importance. The importance of randomness of both the axial and transverse strengths increases with applied stress while the shear strength variability importance decreases.

Table 3-7: Importance Factors for a 60° Gr/Ep Composite

Failure Criterion:	Tsai - Hill			Tsai - Wu		
Applied Stress, MPa	55	60	65	55	60	65
$I_{Lt}$	$\approx 0$	$\approx 0$	$\approx 0$	0.004	0.004	0.005
$(I_{Lc})$	————	————	————	0.008	0.009	0.011
$I_{Tt}$	0.996	0.992	0.986	0.988	0.980	0.968
$(I_{Tc})$	————	————	————	0.017	0.023	0.031
$I_{LTs}$	0.092	0.126	0.167	0.150	0.199	0.250

The importance factors of the variables shown in table 3-7 vary somewhat depending upon the failure criterion employed for the analysis. The variability of the transverse tensile strength dominates the cases analyzed with both the Tsai-Hill failure criterion and the Tsai-Wu failure criterion. The variability of the shear strength also has a modest contribution. The increased importance of the randomness of the transverse strengths is due to the increase in orientation angle. The randomness of the axial strengths has nearly no effect on these cases.

#### 4. SUMMARY AND CONCLUSION

In this report we examined a variety of methods used to evaluate the reliability of composite laminae. The initial portion of this report discussed the necessary steps involved in posing the design problem in the required form. The specific form used the composite strengths as design variables and the Tsai-Hill and Tsai-Wu failure criteria for failure determination. The first method type discussed was the fast probability integration (FPI) methods. These consisted of two groups: first order, second moment reliability methods and second order, second moment reliability methods. The second type discussed was the Monte Carlo methods. The two methods discussed in this category were the basic Monte Carlo and the Importance Sampling methods. These methods were examined in detail and their procedures for implementation were outlined.

The next portion of this report used calculated probability of failure estimates to evaluate these reliability methods. A general comparison of the accuracy of the probability of failure estimates, the efficiency of the methods at determining these estimates, and the conservatism of the estimation was performed. This was designed to compare how these methods performed when analyzing the specific type of design problem used in this report and to describe the general methodology involved in making a comparison. The sensitivity of the probability of failure estimate to variations in the parameters of the distribution of the design variables was discussed. This involved specific sensitivity studies and a description of the methodology required. The final section of this report evaluated importance factors. The importance factors of the variables chosen for the example design problems were found, and the impact of eliminating low importance variables was evaluated.

The conclusions derived in this study come in three parts: those made about the application of the methods discussed, those about the sensitivity analysis results and procedure, and those involving importance factors. The sections below will discuss these topics in terms of their general application and their application to the specific problems outlined within this report.

The Hasofer-Lind method is by far the most efficient method discussed. It is not only the simplest method to implement, but it also requires the least amount of computational time to perform. The limitation of this method would appear to be the relatively coarse approximation of the failure surface. From previous examples (see ref. 5) this appeared to make this method highly inaccurate. From these results, the probability of failure estimates given by this method are as accurate as any of the methods chosen (see tables 3-1 to 3-3). This is most likely due to the typically large radius of curvature (when compared to the reliability coefficient) found in these results. In other applications this may not be the case. Other possible benefits of this method are its insensitivity to the number of variables in the design problem and its ability to handle complex failure criteria. Due to these properties and the simplicity of the implementation procedure, this method is suggested for the initial analysis of any design problem. Steps should be taken, however, to verify the accuracy of the calculated results. This verification procedure will likely require the implementation of one of the higher order FPI (or sampling) methods.

The Central Hypersphere method is as efficient as the Hasofer-Lind method, but offers none of its other benefits. In all cases analyzed, probabilities of failure calculated with this method differed substantially from the "exact" value. This is likely due to the flatness of the failure surface in our design problem. In addition to the inaccuracy of this method it also suffers from a strong sensitivity to the number of variables used in the design problem. The positive aspects of this method are that it is both simple and computationally quick and will calculate results which are always conservative. The amount this method over-estimates the probability of failure is the real concern. In general application, this method may result in a structure which is over-designed. This is both inefficient and, typically, expensive. As a result this method is suggested for determining conservative reliability estimates only while another reliability method should be used as the primary method for analyzing reliability.

The second order methods (both the Rotational Paraboloid and the Non-Central hypersphere) offer some benefits which are not found in first order methods. In dangerous curvature cases these methods can predict probability of failure values which are closer to the actual value than those predicted by either of the first order methods (see tables 3-1 and 3-2). This benefit may be eliminated, however, if the approximated failure surface has a variety of curvatures which differ in both sign and magnitude (see table 3-1, Tsai-Wu criterion). These methods as presented here were limited to analyzing design problems which have dangerous curvatures. In addition to these limitations these methods require additional knowledge of the failure surface and may be sensitive to the radius of curvature chosen for the surface approximation. These methods are also more difficult to implement and require more computational time to perform. These properties limit the application of these methods to specific types of design problems. However, from our data, it is shown that use of the non-central hypersphere approximation with the smallest radius of curvature offers a reliability estimate which is conservative but more accurate than results calculated using the central

hypersphere method. Using these methods for design problems with radii of curvature which are greater than one order of magnitude larger than the reliability coefficient will not offer substantial improvements over a first order method, because the failure surface is so "flat" that linear surface is a good approximation. In general, it should first be determined that the design problem does not have  $R \gg \beta$  before the additional steps involved in this method are performed. These two approximations (Rotational Paraboloid and the Non-Central hypersphere) appear to offer very similar results. From these results it does not appear necessary to analyze a design problem using both approximations.

The Importance Sampling method is far less computationally efficient than any of the FPI methods discussed. It does, however, offer a possible improvement in the probability of failure estimates for design problems with safe curvatures (see table 3-1, Tsai-Wu criterion). This is not possible with FPI methods, because all the methods utilize "dangerous" curvatures to make approximations. Another benefit of this method is that it does not suffer from any intrinsic conservatism as some of the FPI methods do. The probability of failure estimates calculated with this methods have a similar accuracy for all types of design problems. The only negative feature of this method is that it depends upon repetitive simulations. This means that as the probability of failure decreases the amount of required computation time increases. However, if the number of design problems being analyzed is small, the amount of computational time required for each analysis may not be prohibitive. This method is suggested if it has been determined that the FPI methods are giving inaccurate results or if the first or second derivatives of the failure surface are difficult to obtain.

The final method examined in this thesis is the Rackwitz-Fiessler method. This is the only method capable of analyzing design problems with non-normally distributed variables. This method is slightly more difficult to implement than the other first order methods. It does not appear to give accurate probability of failure estimates when compared with the "exact" value. In fact an equivalent design problem using normally distributed variables (which have the same means and standard deviations as the non-normal distributions) appears to be more accurate. The inaccuracy of the Rackwitz-Fiessler method may be due to the type of design problem being used in the analysis, however. The method itself has not been found to have any intrinsic problems which would result in erroneous results. Due to the relatively limited examination of this method only the most conservative approaches can be suggested. The Central Hypersphere approximation is recommended for calculating an upper limit to the probability of failure.

**Sensitivity.**—All of the presented methods can be examined to determine the sensitivity of the calculated probability of failure to changes in the parameters of a design variable's distribution. This procedure could be used to determine the importance of obtaining a good estimate of the variable's distribution. The only problem with the proposed procedure is its case dependence. To determine the sensitivity of a design problem to a specific set of design parameters requires a complete sensitivity study. It is suggested that if only one parameter is to be modified, then a sensitivity study using this method is appropriate; but if multiple parameters are to be modified then this procedure may be too time-consuming to perform.

**Importance Factors.**—The importance factors section of this report offers an approach to simplify design problems. In general the randomness of any variable which has an importance factor approximately equal to zero will have very little effect on the calculated results. Using this information, the problem can be simplified by replacing this variable with its mean value in the failure criterion. In all of the cases analyzed the variability of the shear strength was found to be very important while the variabilities of axial strengths were found to be relatively unimportant (see tables 3-5 to 3-7). The importance of the other strengths' variabilities was dependent upon the applied load, orientation angle and failure criterion used. A variable which was found to be unimportant could be replaced with a deterministic value, which simplifies the problem (see example 3-5a). This procedure may not offer significant computational time savings when using an FPI method, due to the small number of design variables used. Unless a problem could be reduced to a single random variable, which would offer a closed form solution, the savings would be minimal. For simulation (Monte Carlo) methods, however, proportional savings in computer time could be achieved for the random variables eliminated..

## REFERENCES

1. Ang, A.H-S., and Tang, W.H., Probability Concepts in Engineering Planning and Design, Vol. II, John Wiley & Sons, New York, (1984), pp. 333-435.
2. Cederbaum, G., Elishakoff, I. and Librescu, L., "Reliability of Laminated Plates via the First-Order Second-Moment method," *Composite Structures*, **15**, (1990), pp. 161-167.
3. Cederbaum, G., et al., Random Vibrations and Reliability of Composite Structures, Technomic Publishing Co., Lancaster, Pennsylvania, (1992), pp. 107-111.
4. Elishakoff, I. and Hasofer, A.M., "Exact versus Approximate Determination of Structural Reliability," *Applied Science Research*, **44**, (1987), pp. 303-312.
5. Fiessler, B., Neumann, H.-J. and Rackwitz, R., "Quadratic Limit States in Structural Reliability," *Journal of the Engineering Mechanics Division*, **105**, (1979), pp. 661-676.
6. Harbitz, Alf, "An Efficient Sampling Method for Probability of Failure Calculation," *Structural Safety*, **3**, (1986), pp. 109-115.
7. Hasofer, A.M., and Lind, N.C., "Exact and Invariant Second-Moment Code Format," *Journal of the Engineering Mechanics Division*, **100**, (1974), pp. 829-824.
8. IMSL Programs Package, State University of New York at Buffalo, (1990).
9. Kendall, M.G., A Course in the Geometry of  $n$  Dimensions, Charles Griffin & Company Ltd., London, (1961), pp. 15-17.
10. Mallick, P.K., Fiber-Reinforced Composites: Materials Manufacturing and Design, Marcel Dekker, Inc., New York, (1988).
11. Patel, J.K., Kapadia, C.H., and Owen, D.B., Handbook of Statistical Distributions, Marcel Dekker, Inc., New York, (1988).
12. Rackwitz, R., and Fiessler, B., "Structural Reliability Under Combined Random Load Sequences," *Computers and Structures*, **9**, (1978), pp. 489-494.
13. Shinozuka, M., "Basic Analysis of Structural Safety," *Journal of the Structural Division*, **109**, (1983), pp. 721-740.
14. Tsai, S. W., and Hahn, H. T., Introduction to Composite Materials, Technomic Publishing Co., Lancaster, Pennsylvania, (1984).

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